Multifractal Predictability and Forecasts

Starry Night, Van Gogh, 1889,http://www.thelmagazine.com

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RainGain Workshop
Antwerpen, 31/03/2014

Predictability

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- What can we predict?
  - What are the intrinsic predictability limits?
- Necessary to clarify what are they respectively for systems
  – complex only in time (ODE)
  – complex both in space and time (PDE)
- How to reach these predictability limits? (*)

- (*) how to be “operational” without it?

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Lessons from complexity in time?

– brought a wealth of striking results for (finite) nonlinear (ordinary) differential systems:

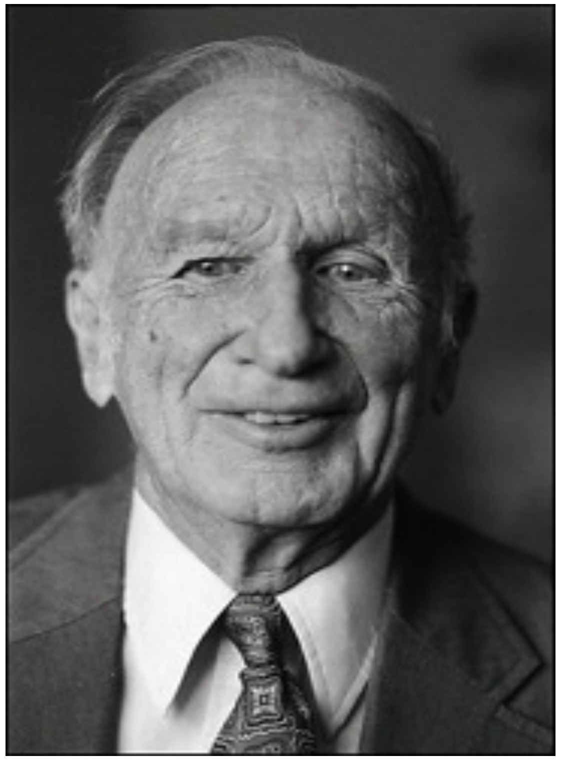
$$\dot{\underline{X}}(t) = \frac{d}{dt}\underline{X} = \underline{F}(\underline{X},\ t)$$

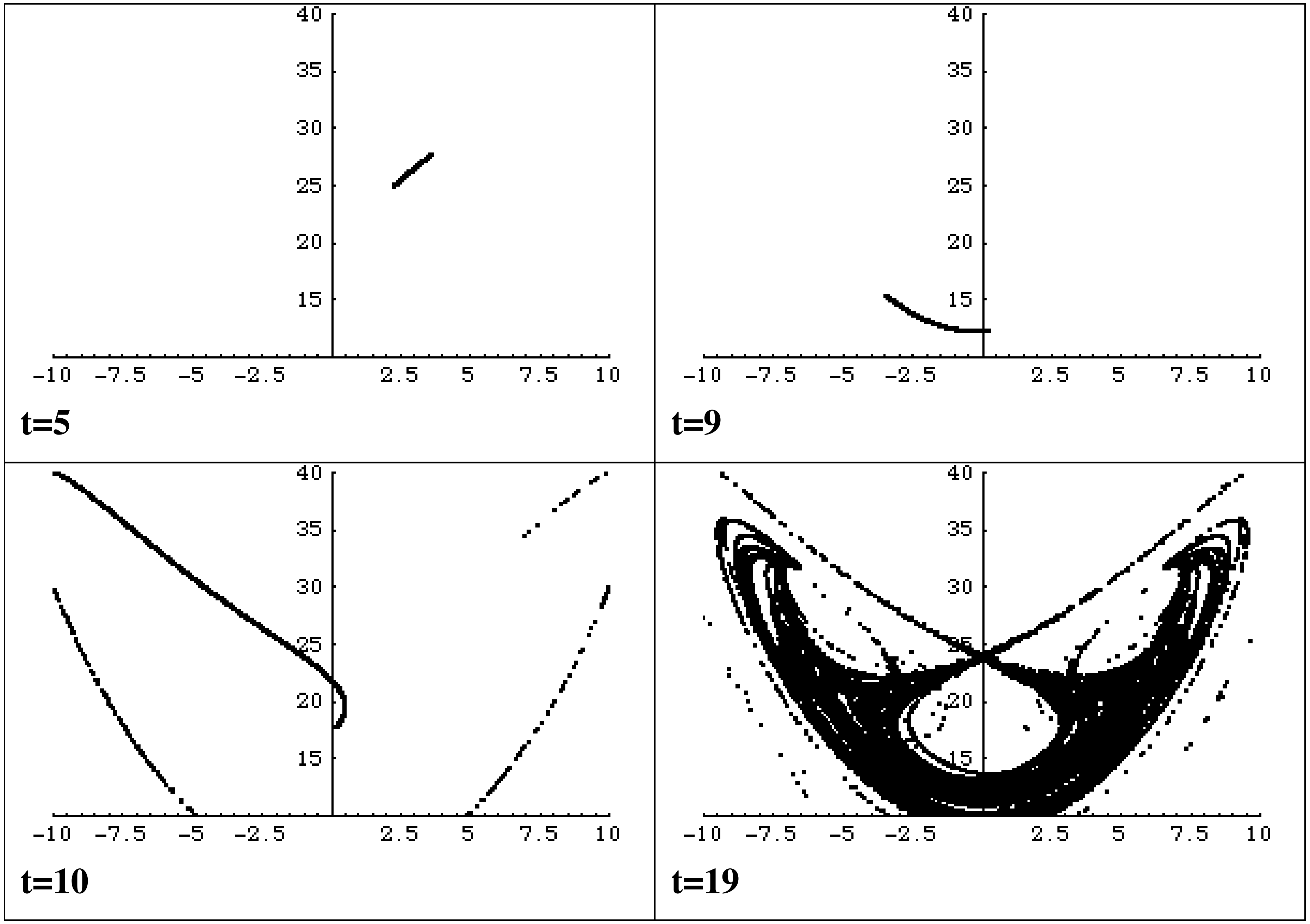
in a d-dimensional embedding space E_d

– or (simpler) iteration maps:

$$\underline{X}_{n+1} = \underline{G}(\underline{X}_n)$$

The Butterfly Effect





Lorenz model: **x-z projection** of the evolution 100,000 points initially uniformly distributed (σ=.027) in the neighborhood of (6.27,13.9,19.5) that quickly spreads over the strange attractor.

Lessons from complexity in time?

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– Multiplicative Ergodic Theorem (M.E.T.)
Lyapunov, 1907; Oseledets, 1968

$$\left|\delta \underline{X}(t)\right| \approx e^{\mu t}\left|\delta \underline{X}(0)\right|$$

– with the (finite) Lyapunov exponent:

$$\mu = < Log^{+}\left(\left\|D_X F\right\|\right) >$$

Hints: pair separation is multiplicatively modulated by the derivative

+ existence of an ergodic measure that defines <.> averages

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Lyapunov exponent of Lorenz model

$$\mu(t) = \int_0^t dt \; Log^+\left(\left\| D_{X_t} F \right\|\right)$$

$$\overline{\mu} \approx 2.33341$$

Lessons from complexity in time?

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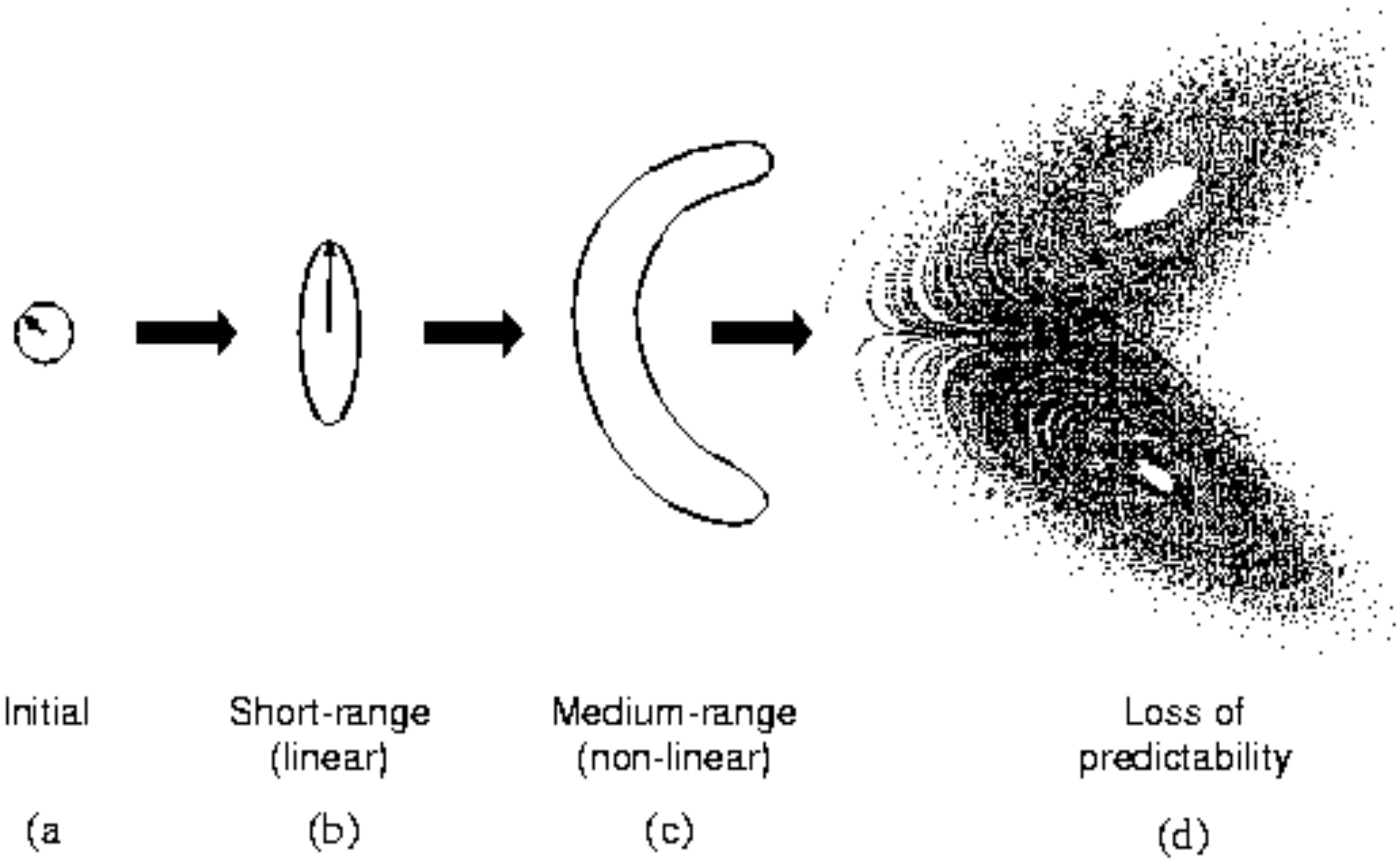
– Liouville equation **(LE)** for intermediate times

(Liouville, 1938) for any well-posed finite *d*-dimensional differential system, an ergodic measure exists and is regular w.r.t. the Lebesque measure $dX_1\, dX_2 \ldots\, dX_d$

$$\frac{\partial}{\partial t}\rho(\underline{X},t)+\sum_{i=1}^{d}\frac{\partial}{\partial X_i}\left[\dot{X}_i(t)\rho(\underline{X},t)\right]=0$$

– i.e. a continuity equation of the density of the ergodic measure in the phase space

Butterfly effect and EPS

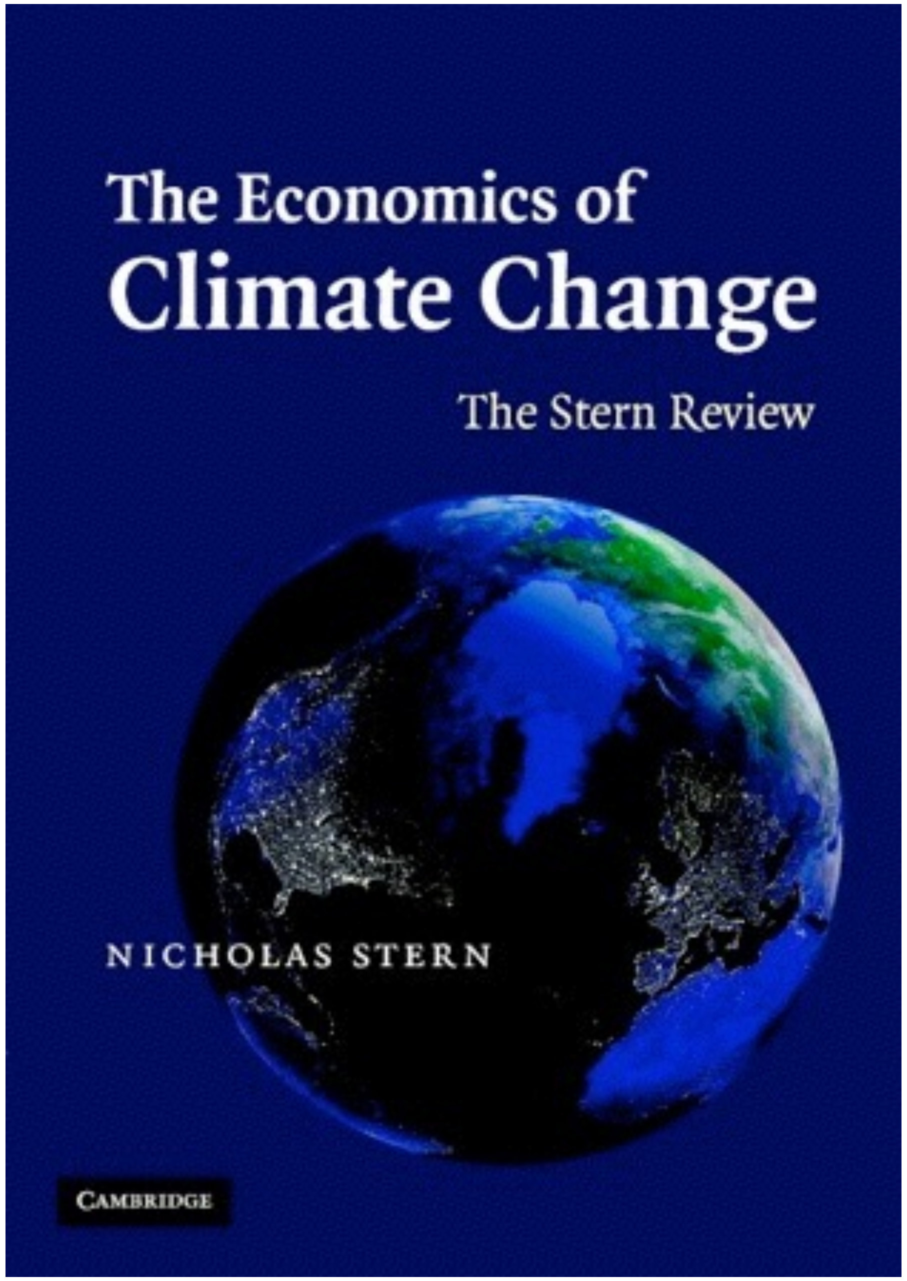


Scheme of the evolution of the empirical pdf evolution of an Ensemble Prediction System (EPS), according to Palmer,1999: from the phase space region occupied by the initial ensemble (a), to (b) linear growth phase, to (c) nonlinear growth phase, to (d) loss of predictability (Palmer, 1993)

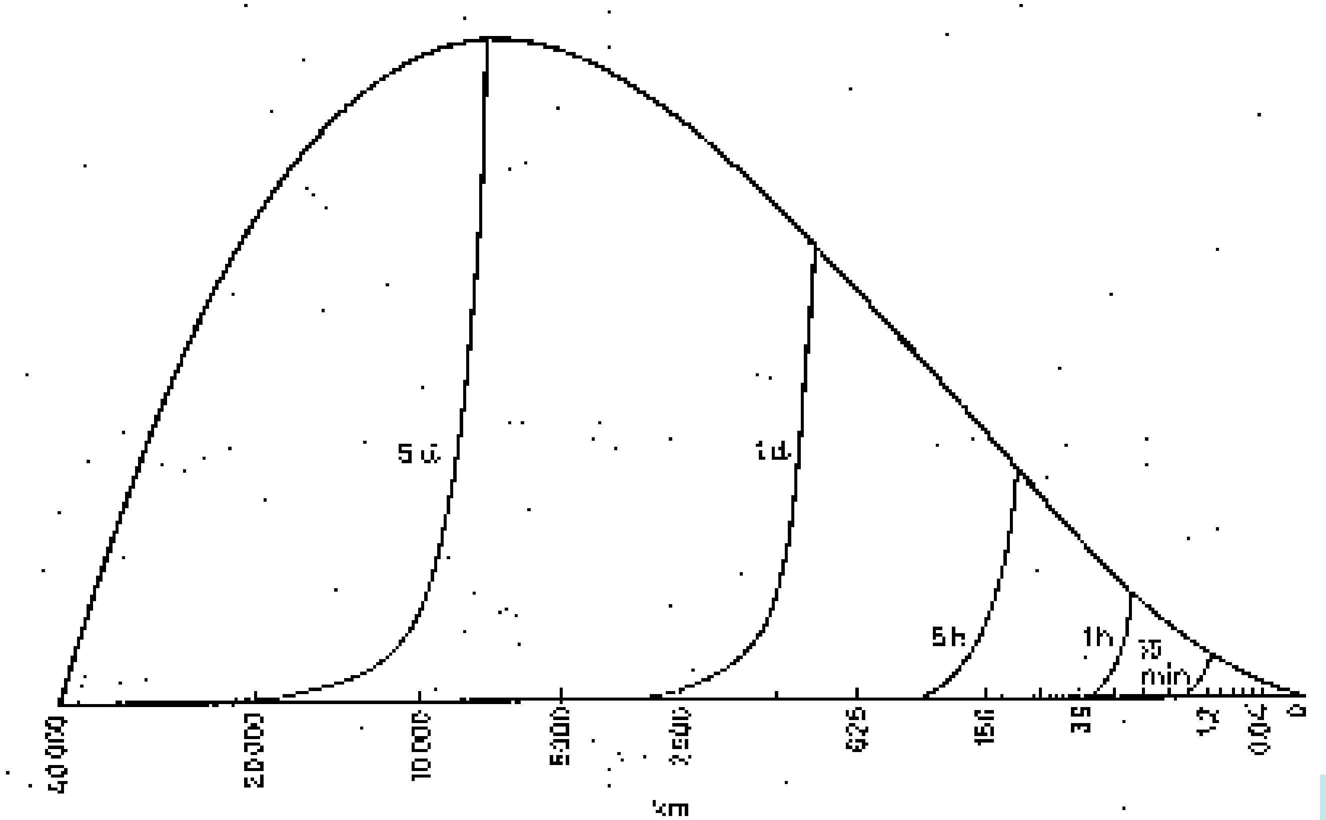
A million dollar problem with trillion-dollar implications!

Is the butterfly effect really true?

Board of Directors and Scientific Advisory Board
Landon T. Clay, Lavinia D. Clay, Finn M.W. Caspersen,
Alain Connes, Edward Witten, Andrew Wiles, Arthur Jaffe
(not present: Randolph R. Hearst III and David R. Stone)



Spectral analysis and closures



$$\ell_c = 1/k_c \approx t^{3/2}$$

Lorenz (1969)
Leith and Kraichnan(1972),
Metais and Lesieur (1986)

Flux from correlated e^C
to
decorrelated energy e^Δ

$$e^c(\underline{x},t) = \underline{u}^2(\underline{x},t).\underline{u}^1(\underline{x},t)$$

$$e^\Delta(\underline{x},t) = \tfrac{1}{2}\left(\underline{u}^2(\underline{x},t) - \underline{u}^1(\underline{x},t)\right)^2$$

Similar results with turbulence phenomenology:

$$\ell_c^{2/3} = \bar{\varepsilon}^{1/3} t^{3/2};\ \bar{\varepsilon} = 10^{-3}\mathrm{m}^2\mathrm{s}^{-3}, \eta \approx 10^{-3}\mathrm{m};$$

Turbulence phenomenology

The **eddy turn-over time** is the characteristic time, if it exists, for structures of scale ℓ to "turn over" within a velocity shear $\delta u(\ell)$: $t \approx \ell / \delta u(\ell)$

It is also the proportional to their time-life (Robinson, 1971) therefore the rate of energy transfer to smaller scales is:

$$\varepsilon(\ell) \approx \delta u^2(\ell) / \tau(\ell) \approx \delta u^3(\ell) / \ell$$

Assuming scale invariance of the energy flux (K41): $\varepsilon(\ell) \approx \bar{\varepsilon}$

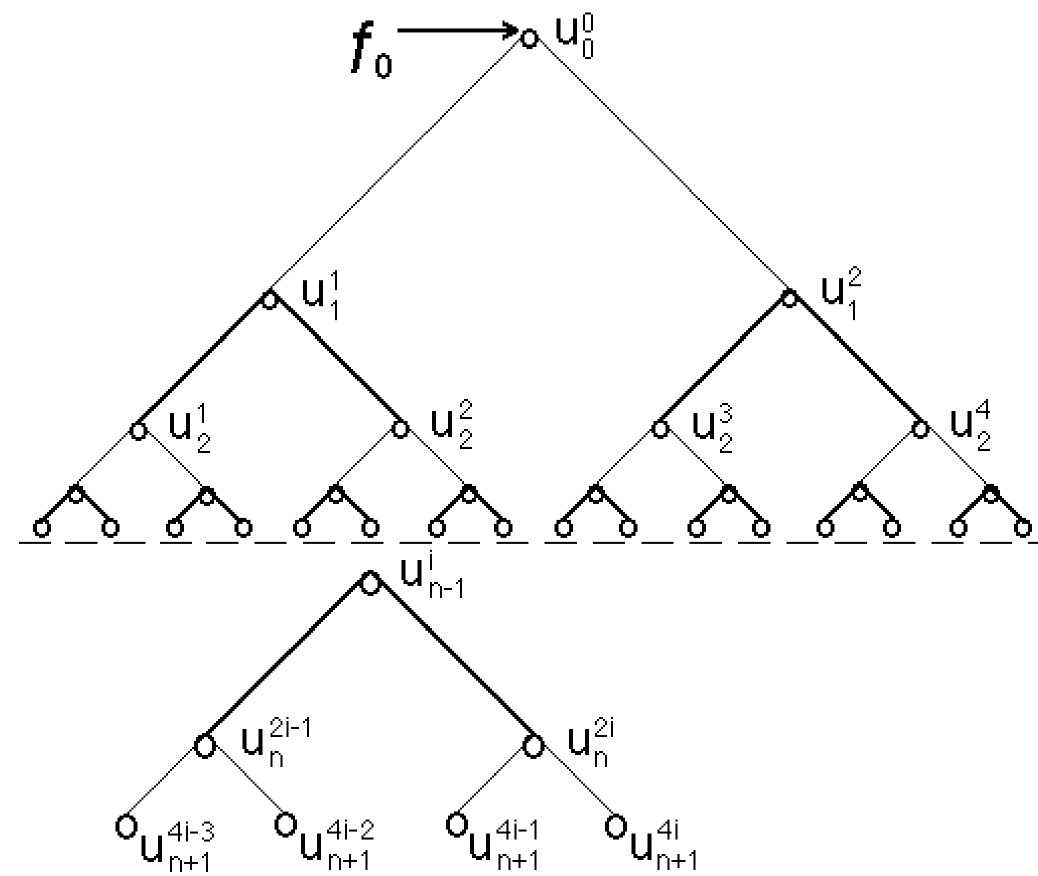
$\mu(\ell) \propto 1/\bar{\tau}(\ell) \propto \bar{\varepsilon}^{1/3} \ell^{-2/3}$ Small scale divergence !

$\ell_e(t) \propto \bar{\varepsilon}^{1/2} t^{3/2}$ uncertainties grow up to larger scales according to a power-law!

How to generalize turbulent phenomenology?

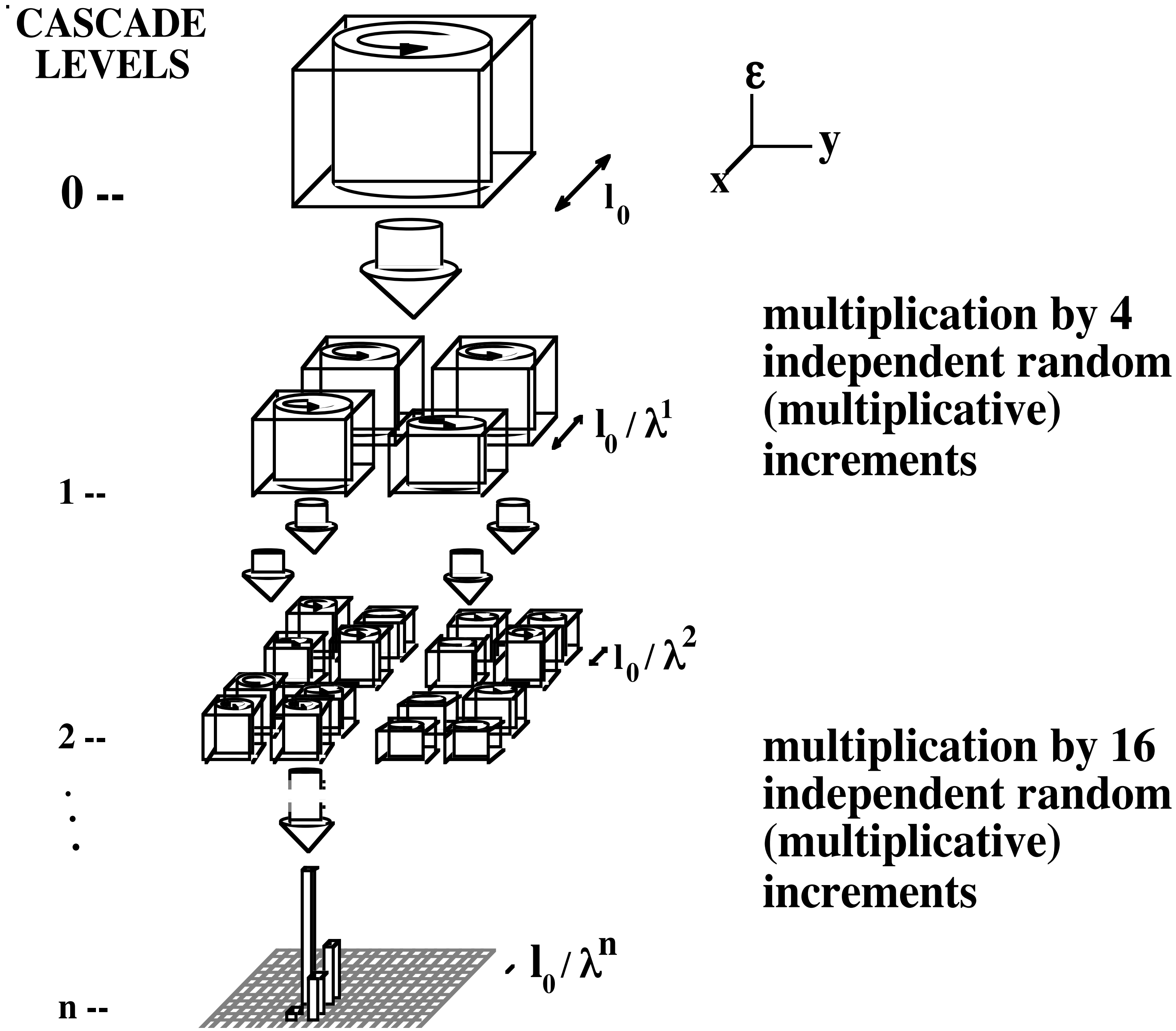
- How to include:
  - intermittency, :
    - strongly non gaussian statistics,
  - scaling anisotropies:
    - Time vs. space:
    - Vertical vs. horizontal:
      - atmosphere is neither 3D, nor 2D, but rather 23/9D ?
- Use (anisotropic) time-space cascades :
  - Outcome: multifractals in the framework of GSI

Hierarchy of space-time structures



Pedagogy: multiplicative cascades

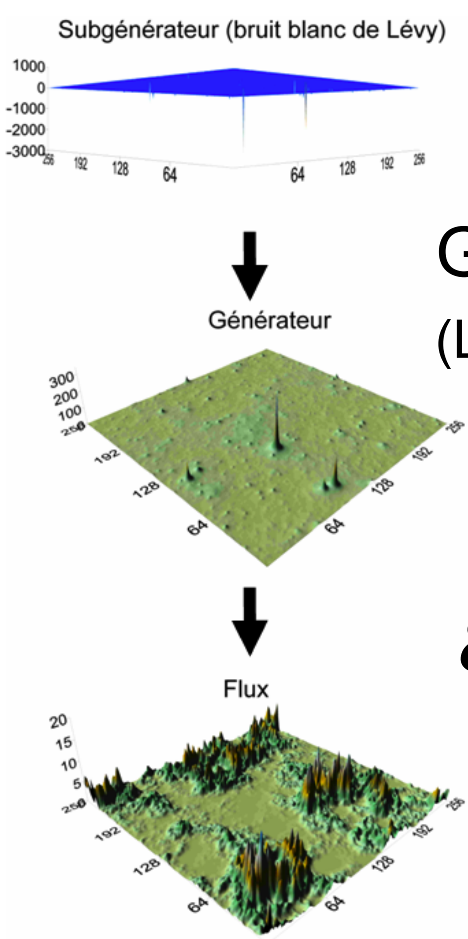
- Richardson poem:
Big whorls have little whorls...
- discrete multiplicative cascade processes (Yaglom 1966, Mandelbrot 1974…)
- from dead/alive alternative (β-model)
- to weak/strong infinite hierarchy of intensities
- supported by an infinite hierarchy of fractals,
- i.e. these fields are in general **MULTIFRACTA**L
- DISCRETE CASCADES are mostly for PEDAGOGY !
  - multiplicative processes are not indispensable !
  - no causality !



(S & L, 1987)

From discrete to continuous and universal cascades

Hint: multiplicative process=exponential of an additive process, but with a small scale singular limit !



Sub-generator $\gamma_{0,\lambda}$ white-noise
(UM: Lévy-noise, index α, β=-1)

Generator Γ_λ:
(Logλ divergence)

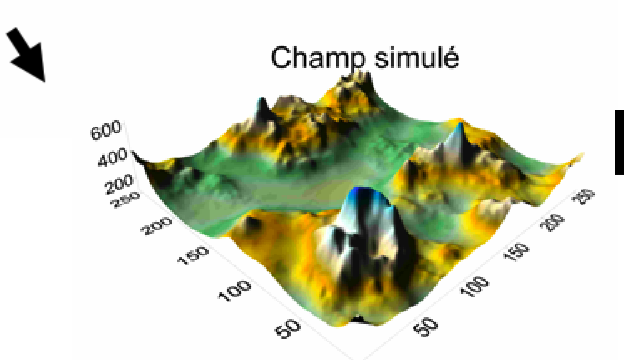
$$-(-\Delta)^{\beta}(\Gamma_\lambda) = \gamma_\lambda,\ \beta = D_{el}/2\alpha',$$
$$1/\alpha + 1/\alpha' = 1$$

$$d\varepsilon_\lambda = \varepsilon_\lambda d\gamma_\lambda$$

ε_λ

$$d\gamma_\lambda - d\Gamma_\lambda = \frac{var(\alpha)d\lambda}{\lambda};\ var(\alpha) = \frac{C_1}{\alpha - 1}$$

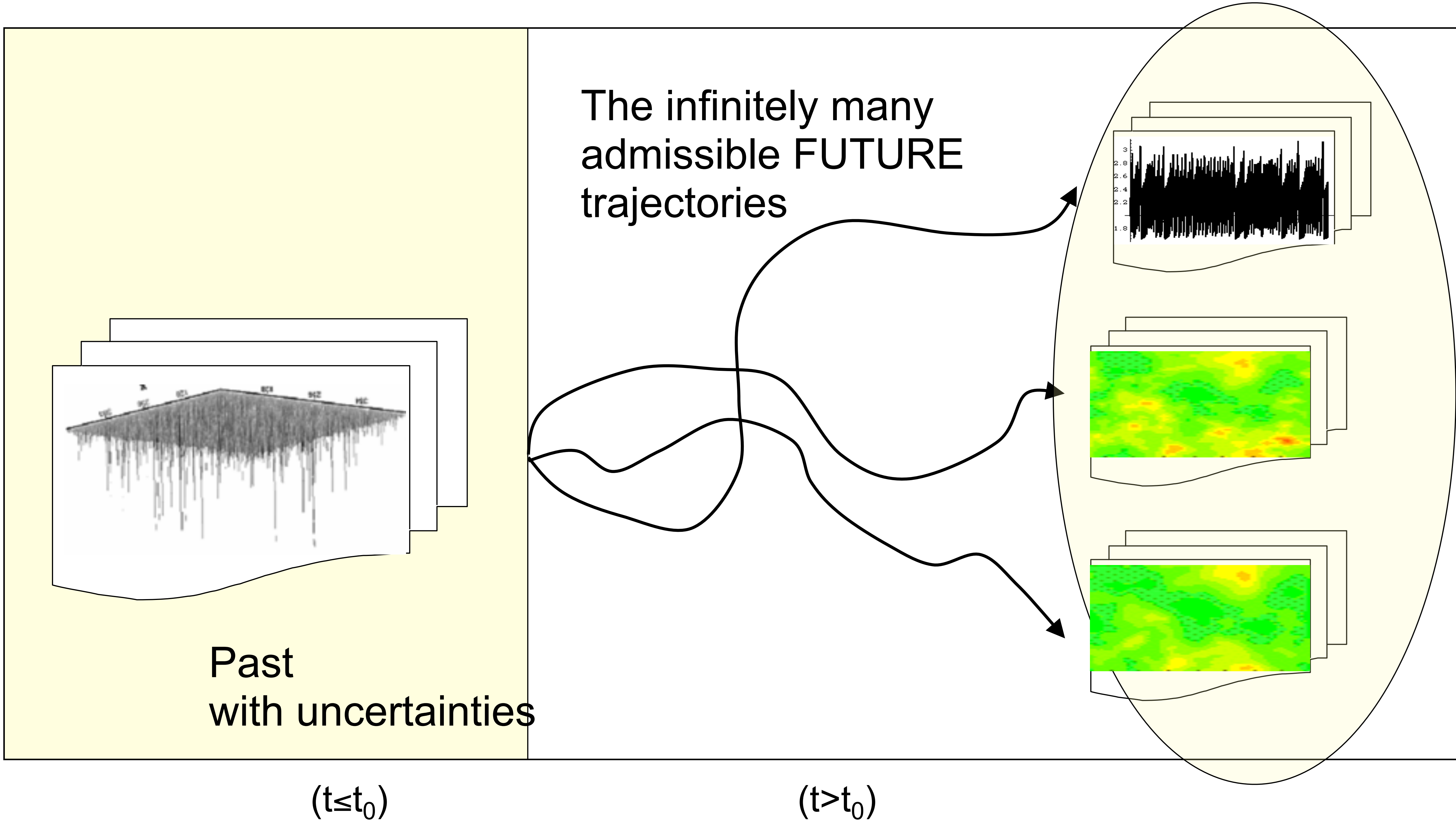
$$-(-\Delta)^{\beta'}(\rho_\lambda) = \varepsilon_\lambda$$
$$\beta' = (D_{el} - H)/2$$



Field ρ_λ

U.M.= stable fixed points of a multiplicative CLT

Forecasts and past memory



Multifractal Predictability

Crude idea:

relaxation of (common) past structures ==> flux of the past

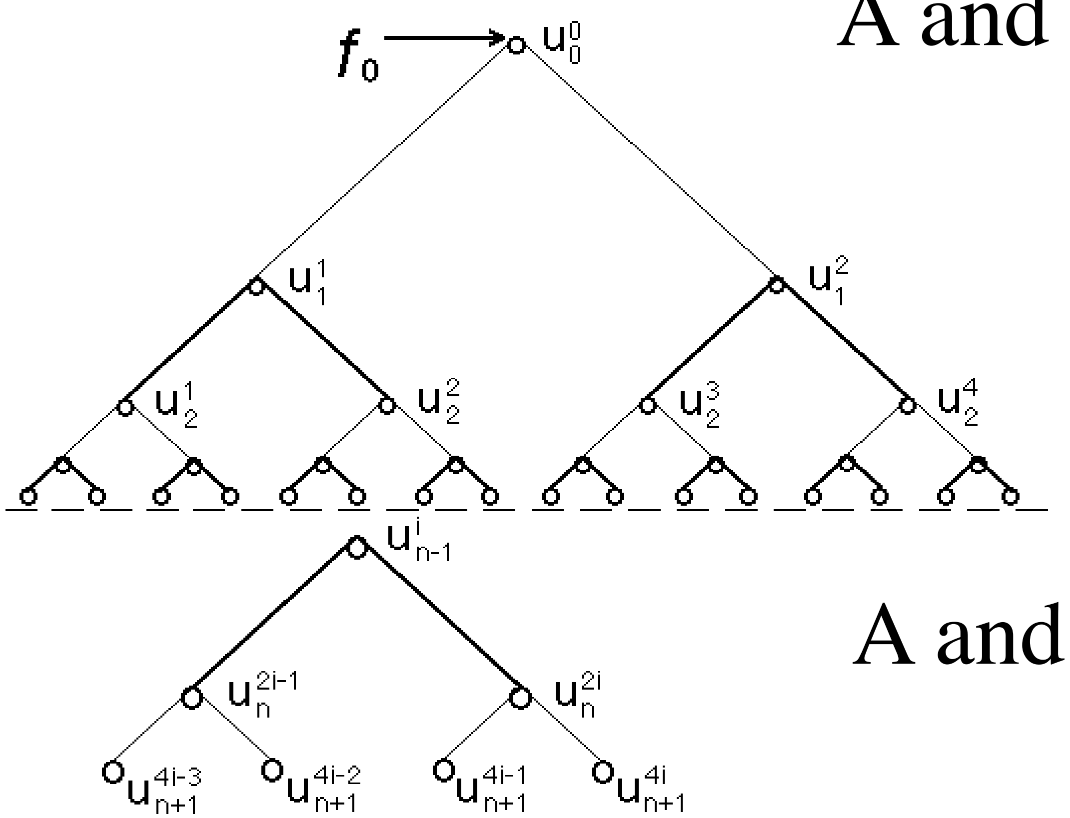
(new) independent structures ==> flux of the future

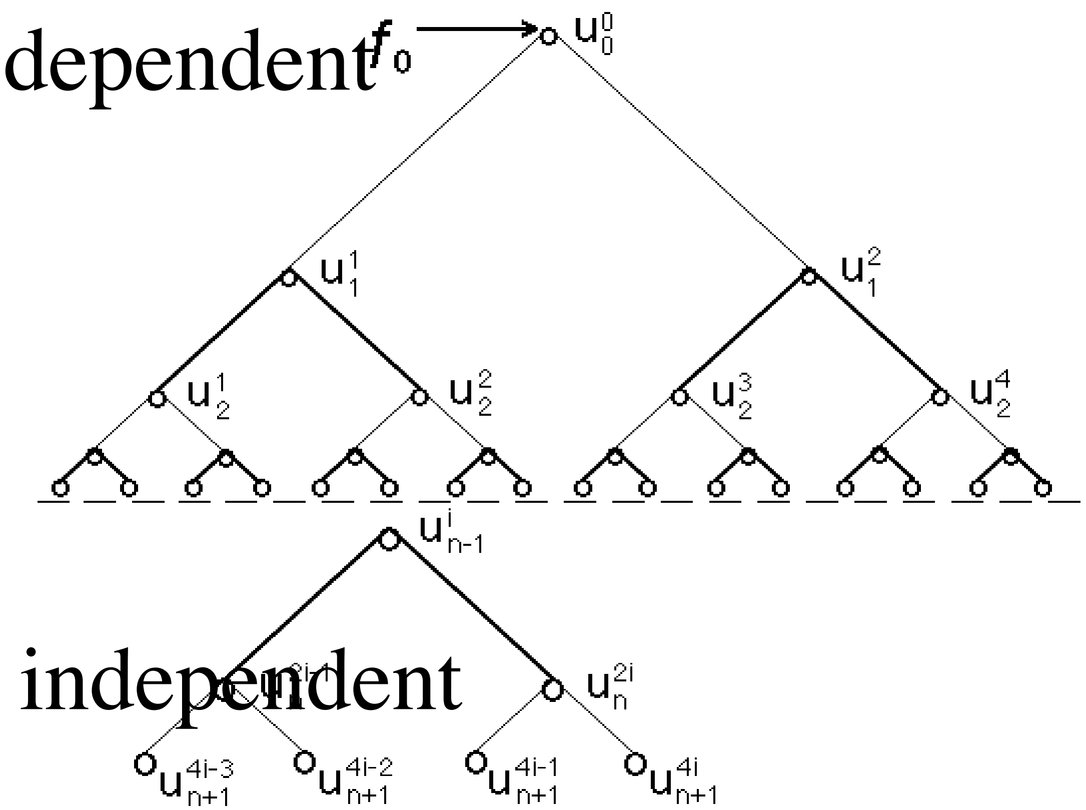
L

Cascade A

Cascade B

A and B strongly dependent





$L/\lambda(t)$

A and B strongly independent

L/Λ

Multifractal predictability

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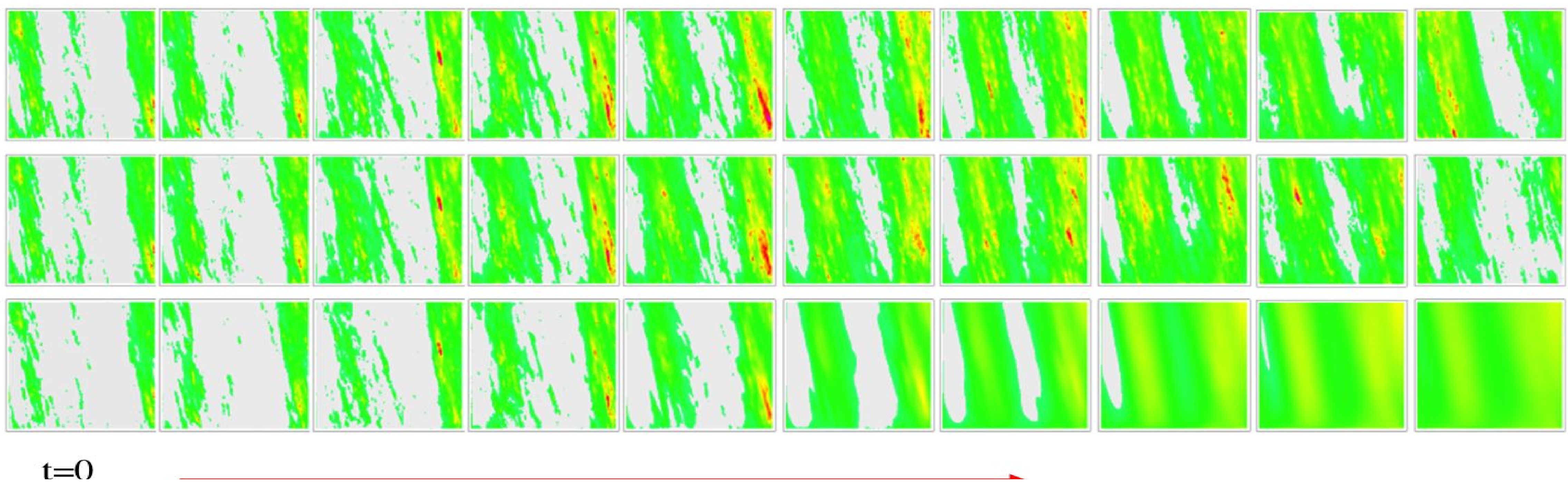
Rain simulation (α=1.5, C_1=0.2, H=0.1 on log scale. Realizations A, B are identical until t=0, then they diverge.

Top: Realization A. Middle: Realization B. Bottom, forecast



•**Power law divergence** between the realizations A and B,
=> **irrelevance** of the **finite** dimensional **'LE + MET' scenario** !
•**Drastic loss of variability of forecast C** with deterministic sub-grid modeling (based on the conservation of the flux) => 'baby theorem': **stochastic sub-grid modeling** does much better than deterministic one!

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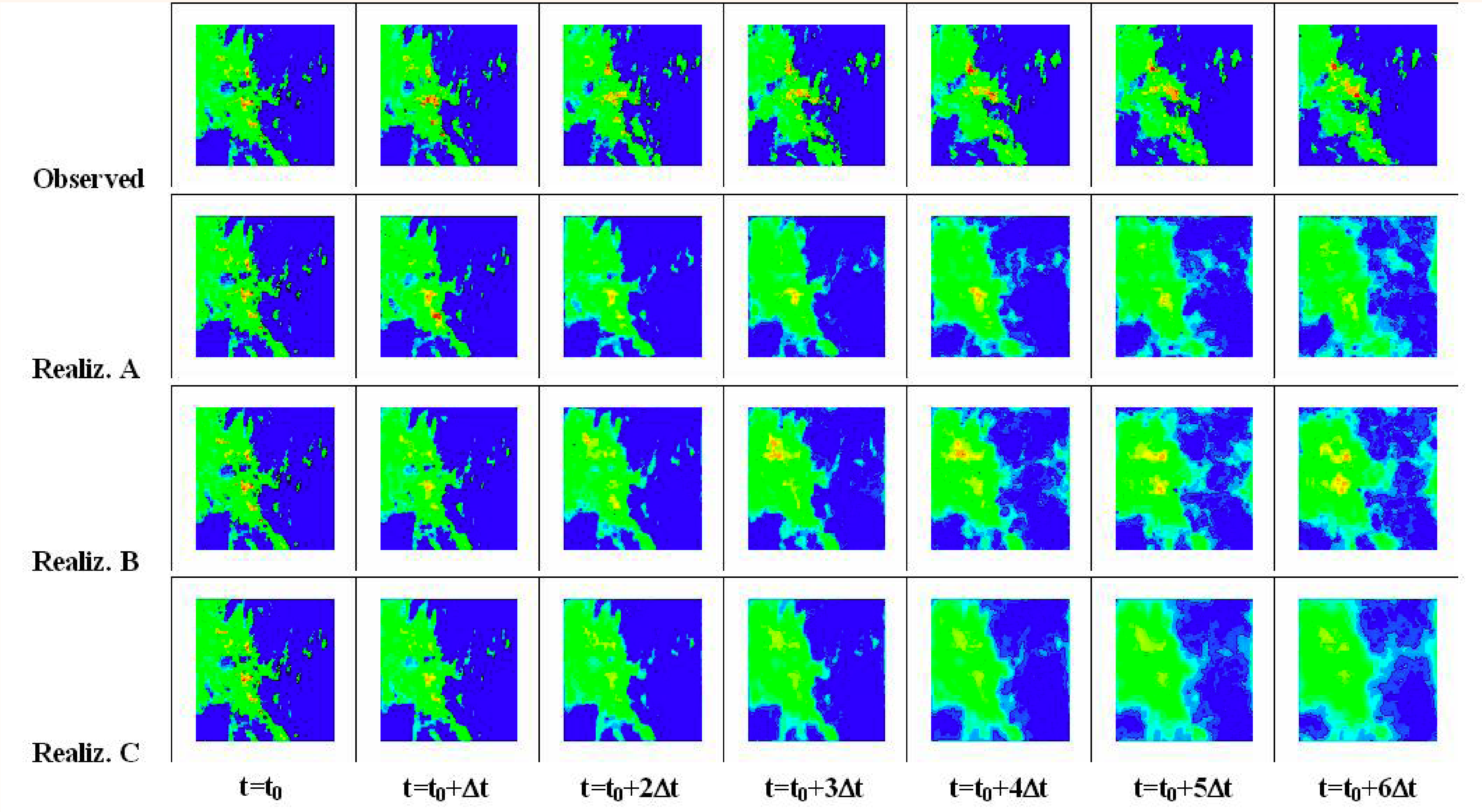
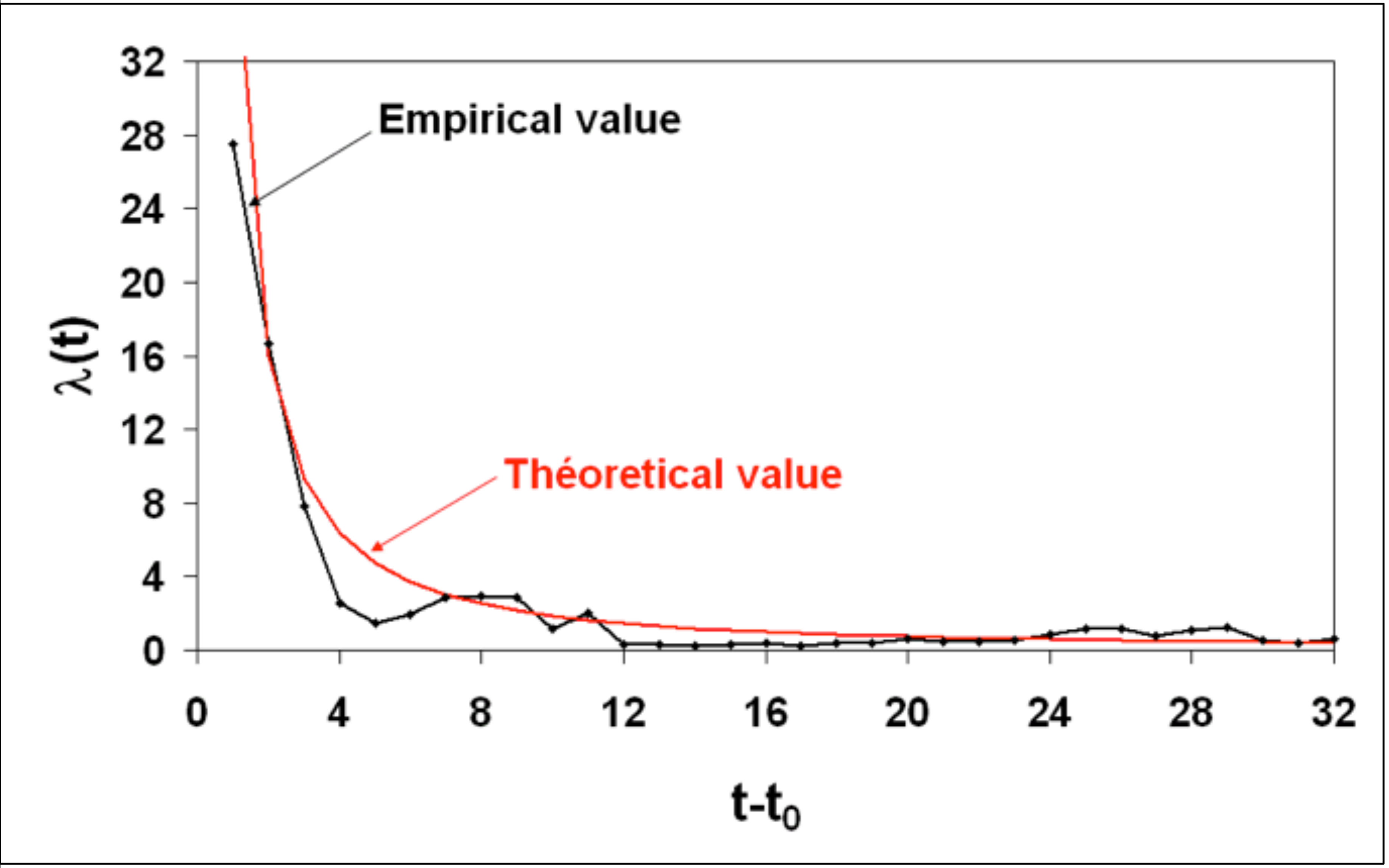
(Schertzer and Lovejoy,Physica A 2004)

Forecasts based on radar data



Power law decay of the common scale ratio



Power law decay of the scale ratio $\lambda(t)$ on which A and B are strongly dependent

STEPS

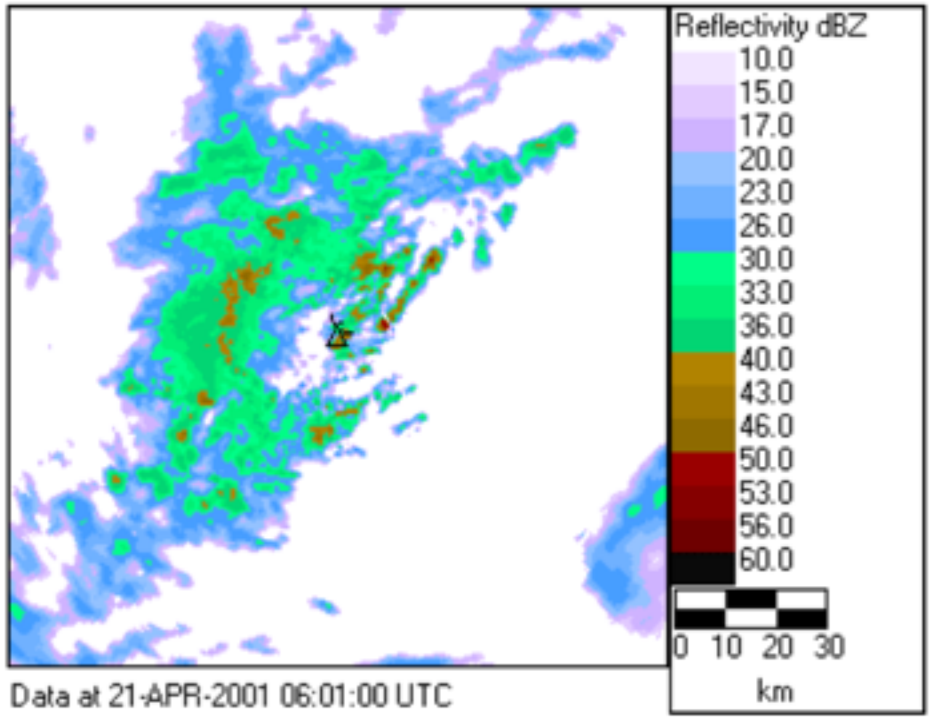


Figure 8 Field of radar reflectivity. The field is from Melbourne, Australia, and is a 256 x 256 km image with 1 km resolution.

STEP is a BOM operational product based on a simplification of multifractal forecasts

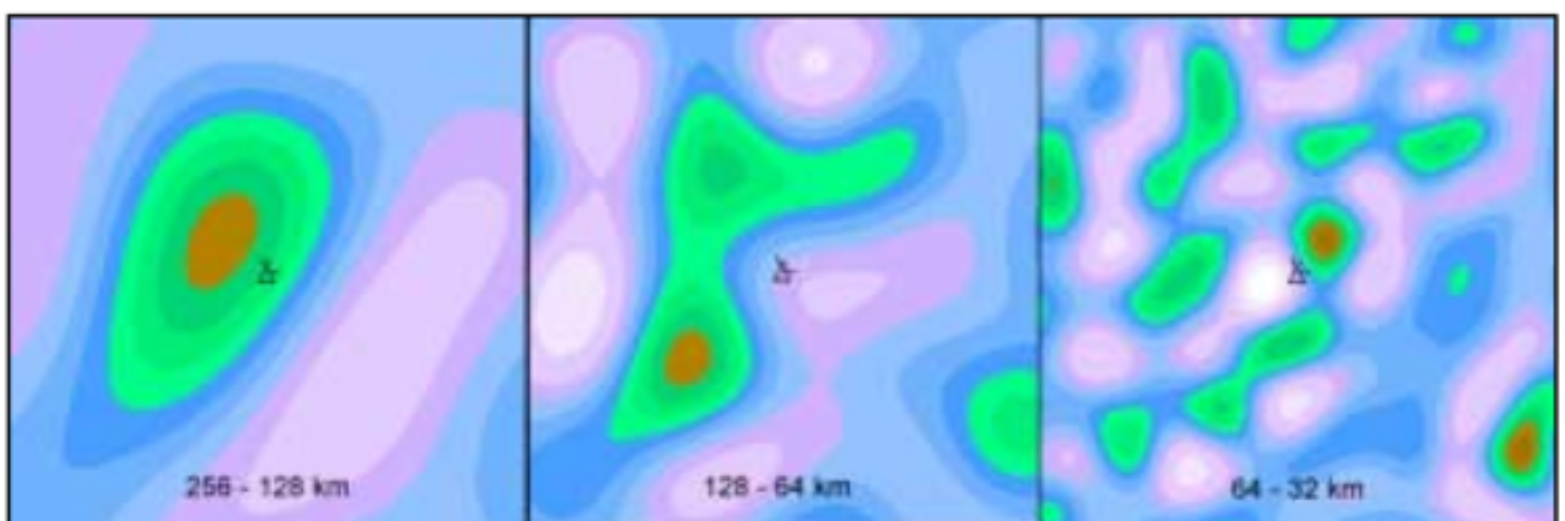


Figure 9 The first three spectral components of the field in Figure 8

(Seed, 2009)

STEPS

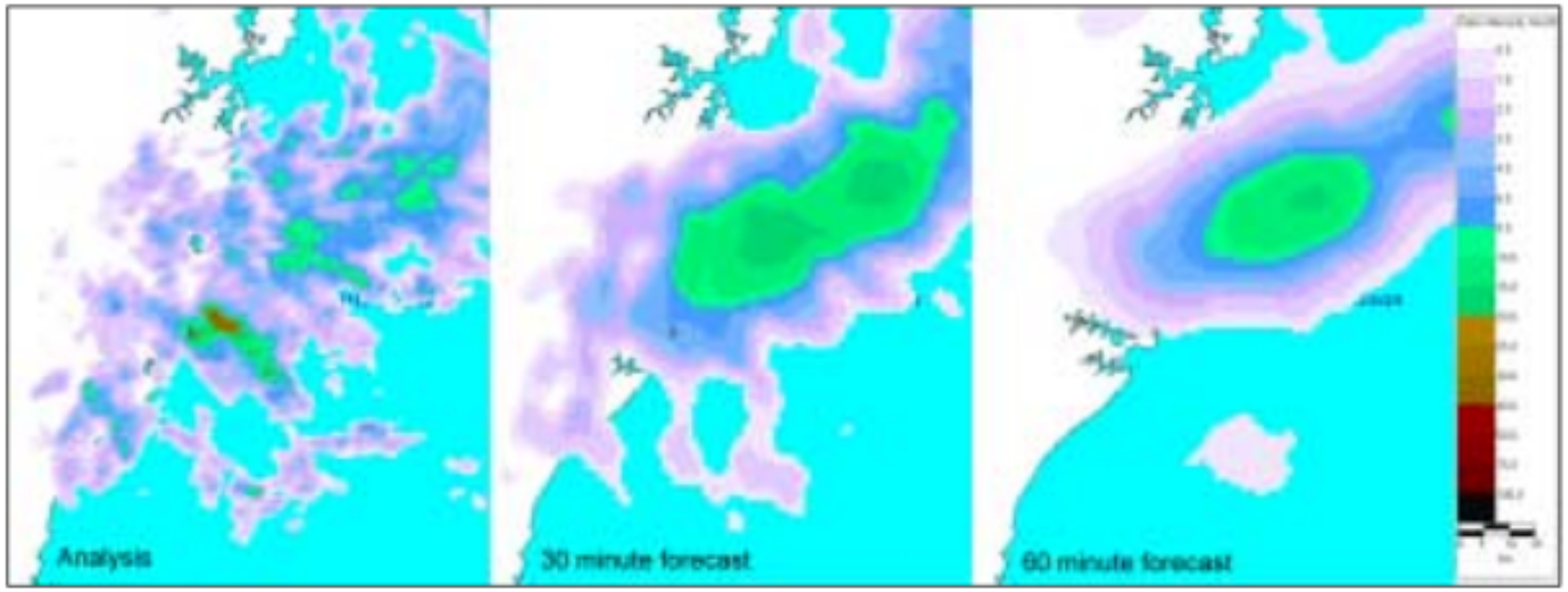


Figure 10 Analysed rainfall, 30- and 60-minute deterministic forecasts of rainfall for the Sydney area for 12:15 12 May 2003

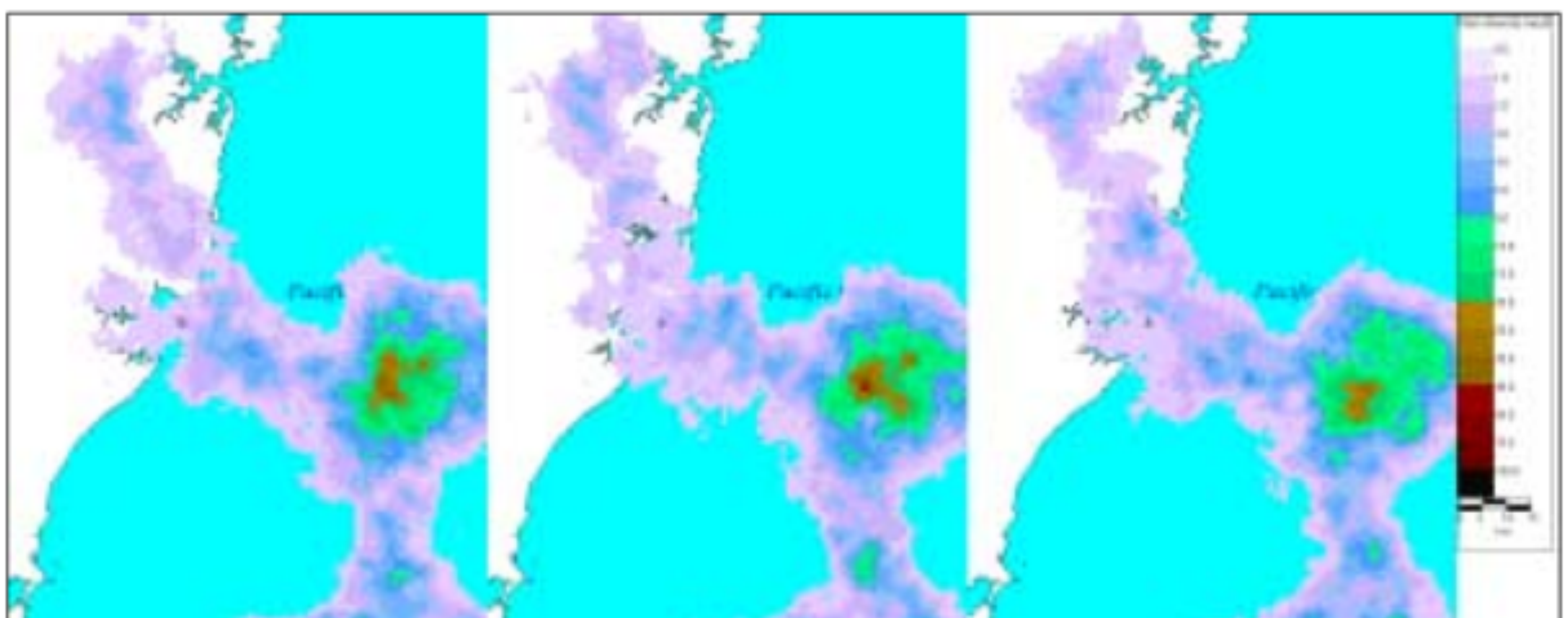
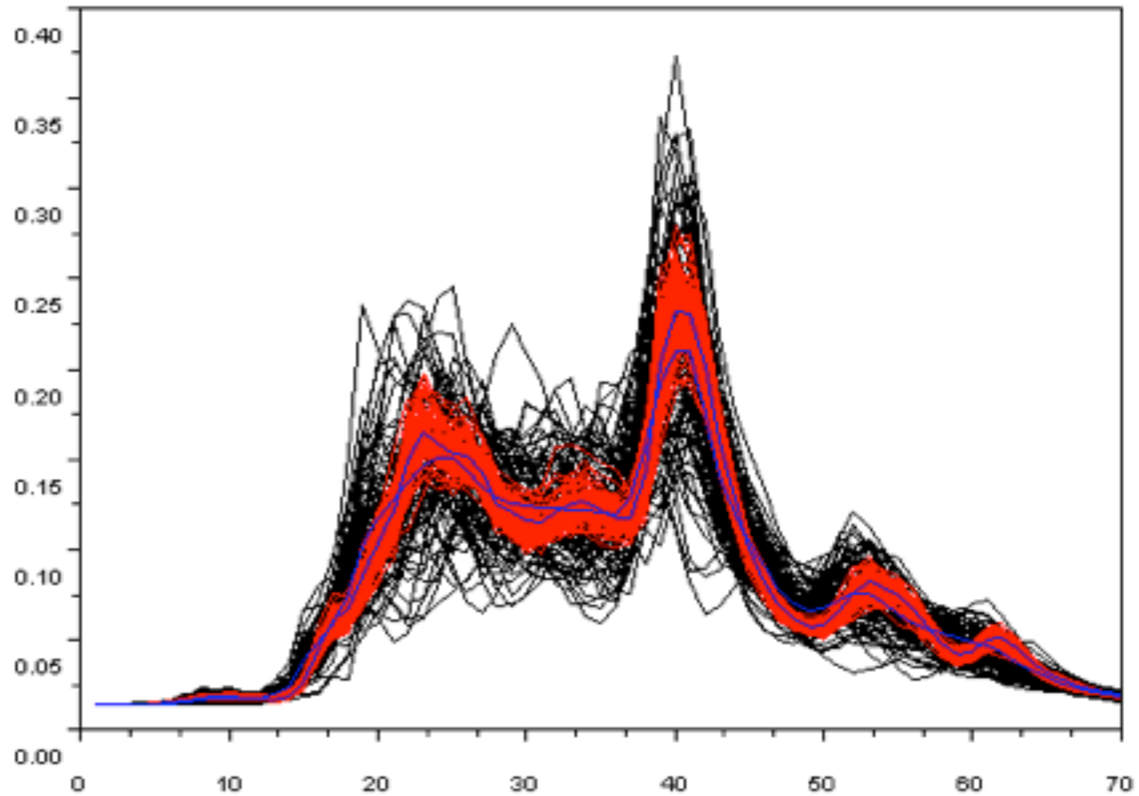


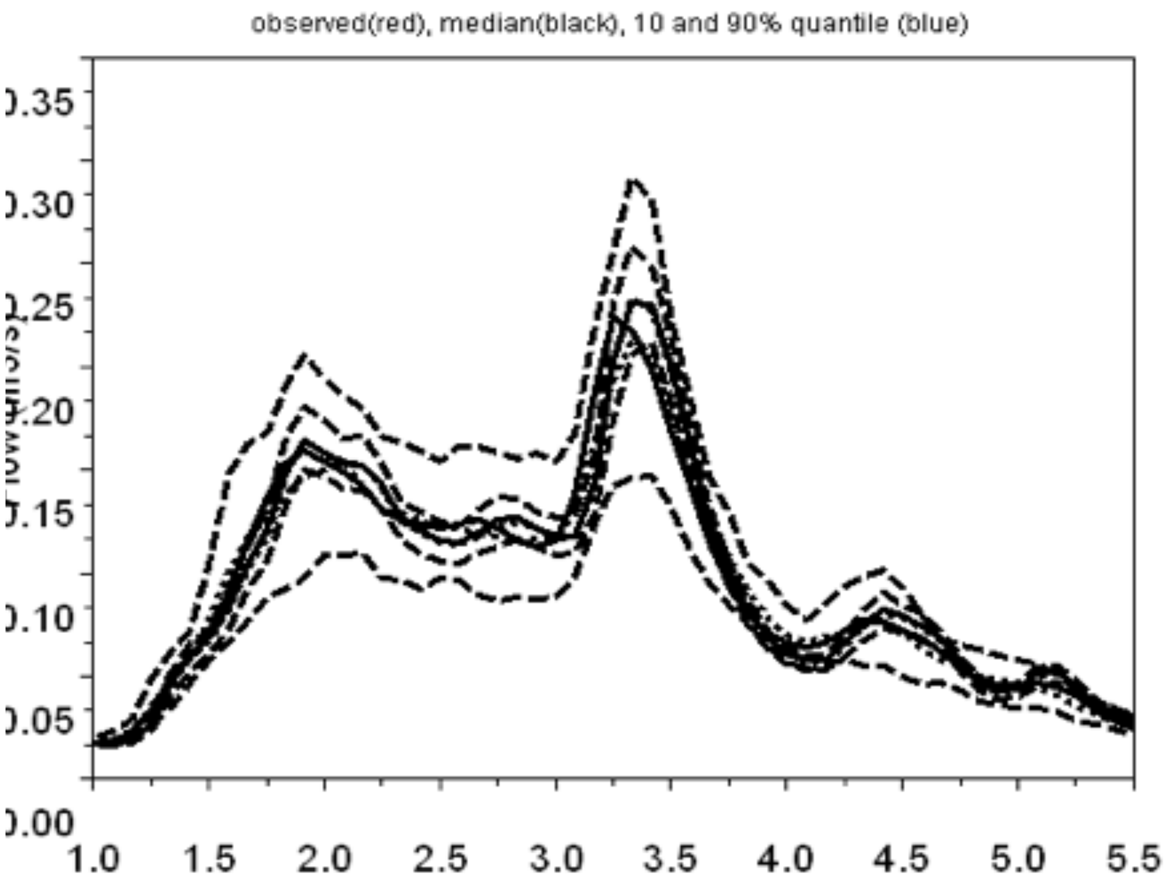
Figure 11 An ensemble of three 1-hour stochastic nowcasts of rainfall over Sydney.

(Seed, 2009)

Small scale variability

Stochastic ensemble (1000 realisations) of multifractal downscaling





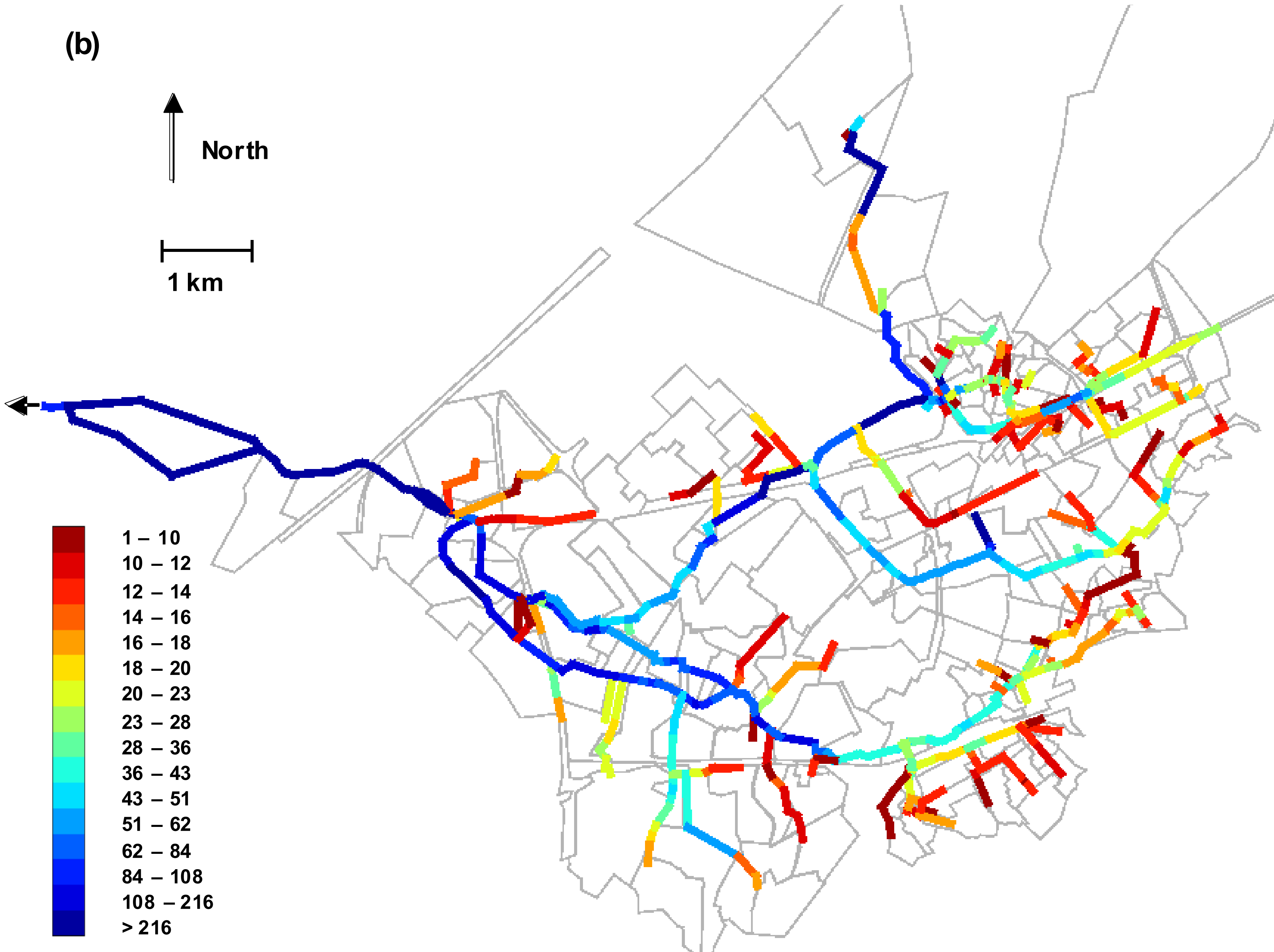
1000 downscaling realisations: 1km -> 125 m — Q_{radar} (1 km)

1000 downscaling realisations: 8 km -> 125 m - - - - $Q_{0.9}$ et $Q_{0.1}$ (125 m)

(Gires, eta l., 2010)

Small scale variability

Stochastic ensemble (1000 realisations) of multifractal downscaling



Estimates of peakflow quantile increases due to small scale variability of the rainfall of in a sewer system in Paris region

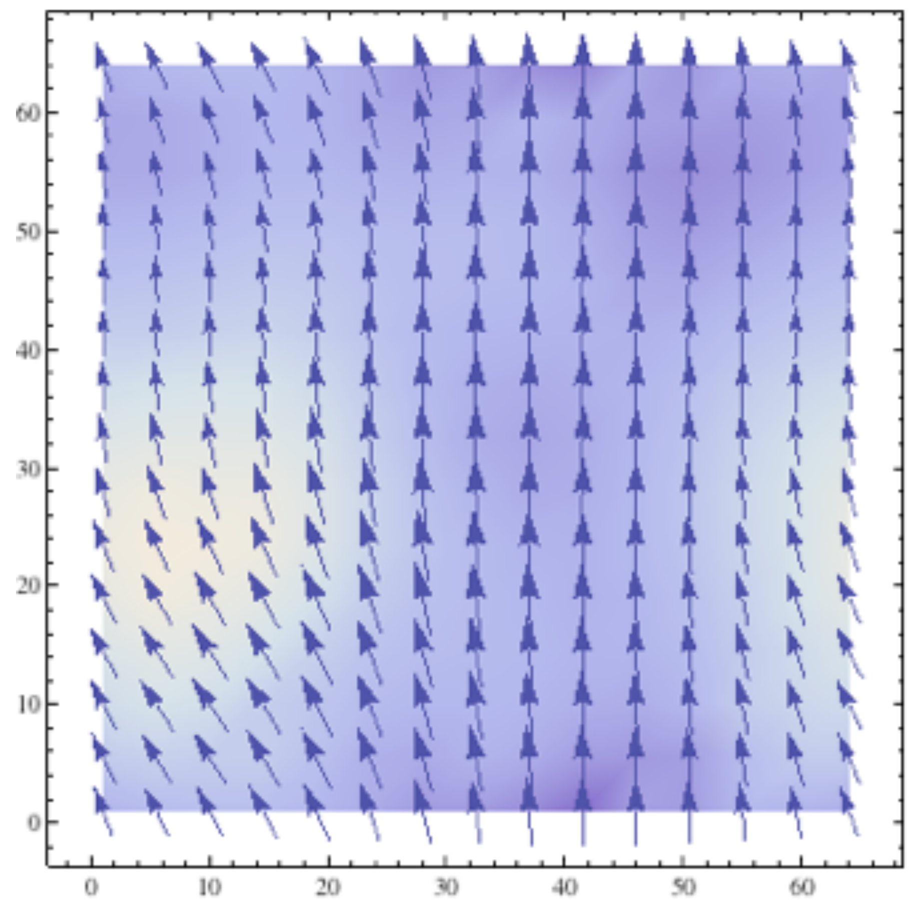
Gires, eta l., 2011)

Surface layer complexity!

explOratorium®

Art piece ‘Windswept’ (Ch. Sowers, 2012): 612 freely rotating wind direction indicators to help a large public to understand the complexity of environment near the Earth surface

WAUDIT Wind resource assessment audit and standardization



Multifractal FIF simulation (S et al., 2013) of a 2D+1 cut of wind and its vorticity (color). This stochastic model has only a few parameters that are physically meaningful.

Both movies illustrate the challenge of the near surface wind that plays a key role in the heterogeneity of the precipitations... and wind energy!

Conclusions

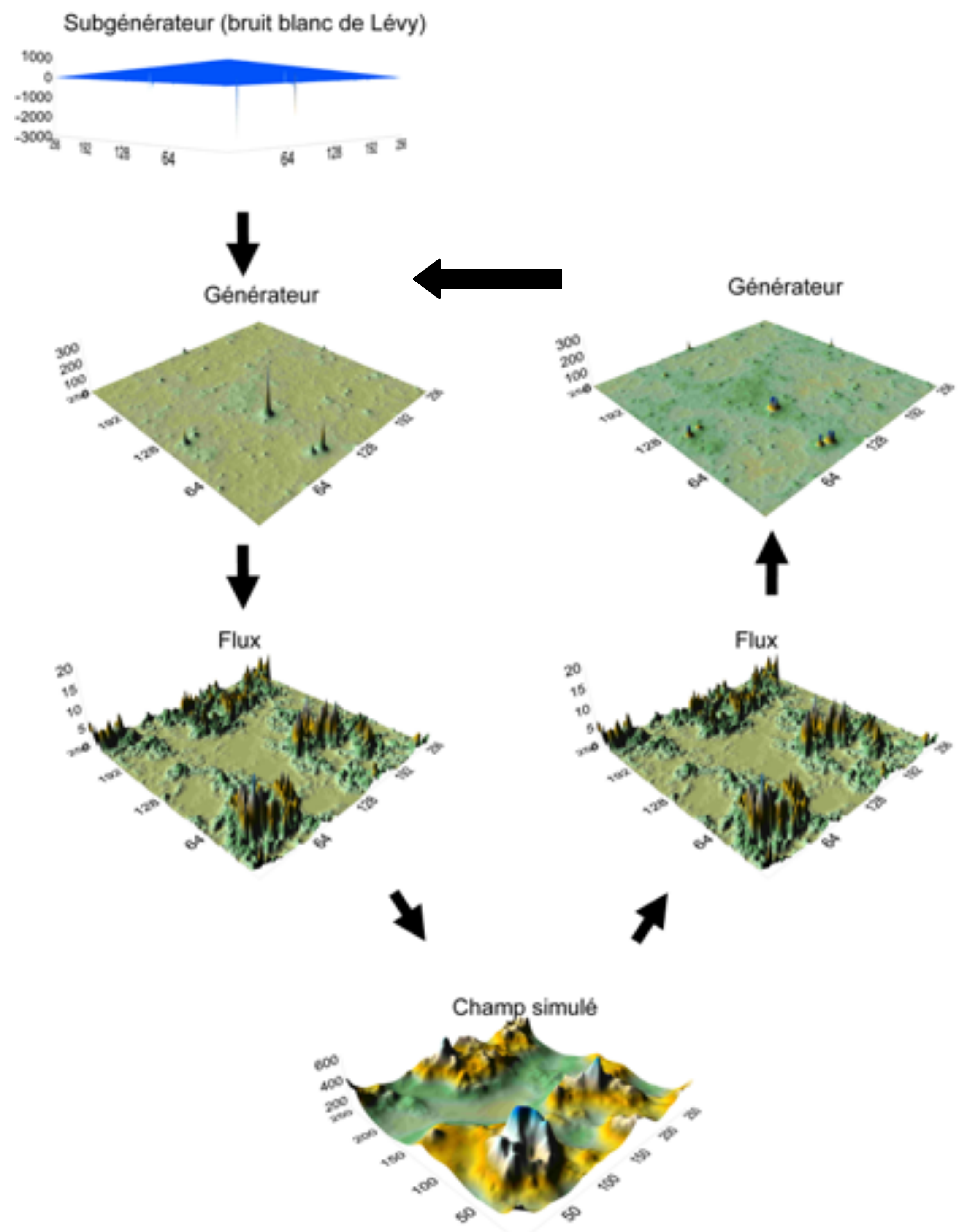
- Multifractal/cascade processes
  - not only help to clarify the predictability of space-time complex systems,
  - but yield concrete methods to dynamically forecast within this predictability limit by:
    - exploiting the past memory
    - yielding admissible futures
  - already generated an operational product (STEPS)
- still interesting/complex problems, e.g.:
    - how to accurately estimate the past generator from real data (deconvolution)
      - therefore from 'imperfect' data
    - what about wind field (added value)?
    - Fokker-Planck equation for MF processes?

- Funding: UE Alban Program, UE FLOODSITE, CNRS/PNRH, RainGain

Forecasts and past memory

How to compute
a possible FUTURE
outcome

Illustration of
a continuous
cascade simulation
from the
subgenerator (white
noise)
to the field
(fractionnaly
Integrated Flux)



How to use
the memory of
the PAST

Illustration of
the ‘deconvolution’
of past data to extract
the past generator from
the observed field

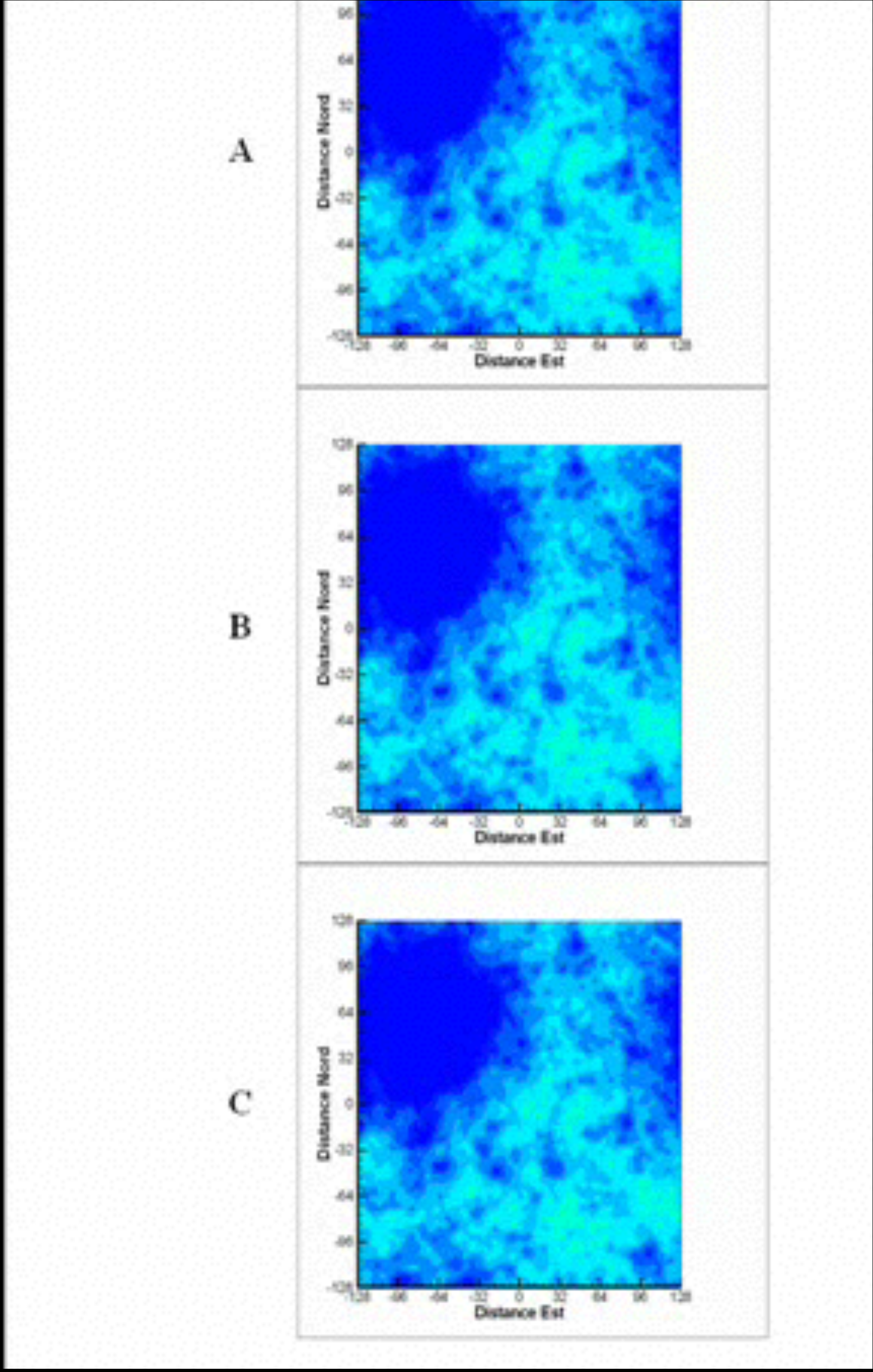
FORECAST: Combine the two generators to get the total flux and the total field

Examples of forecasts

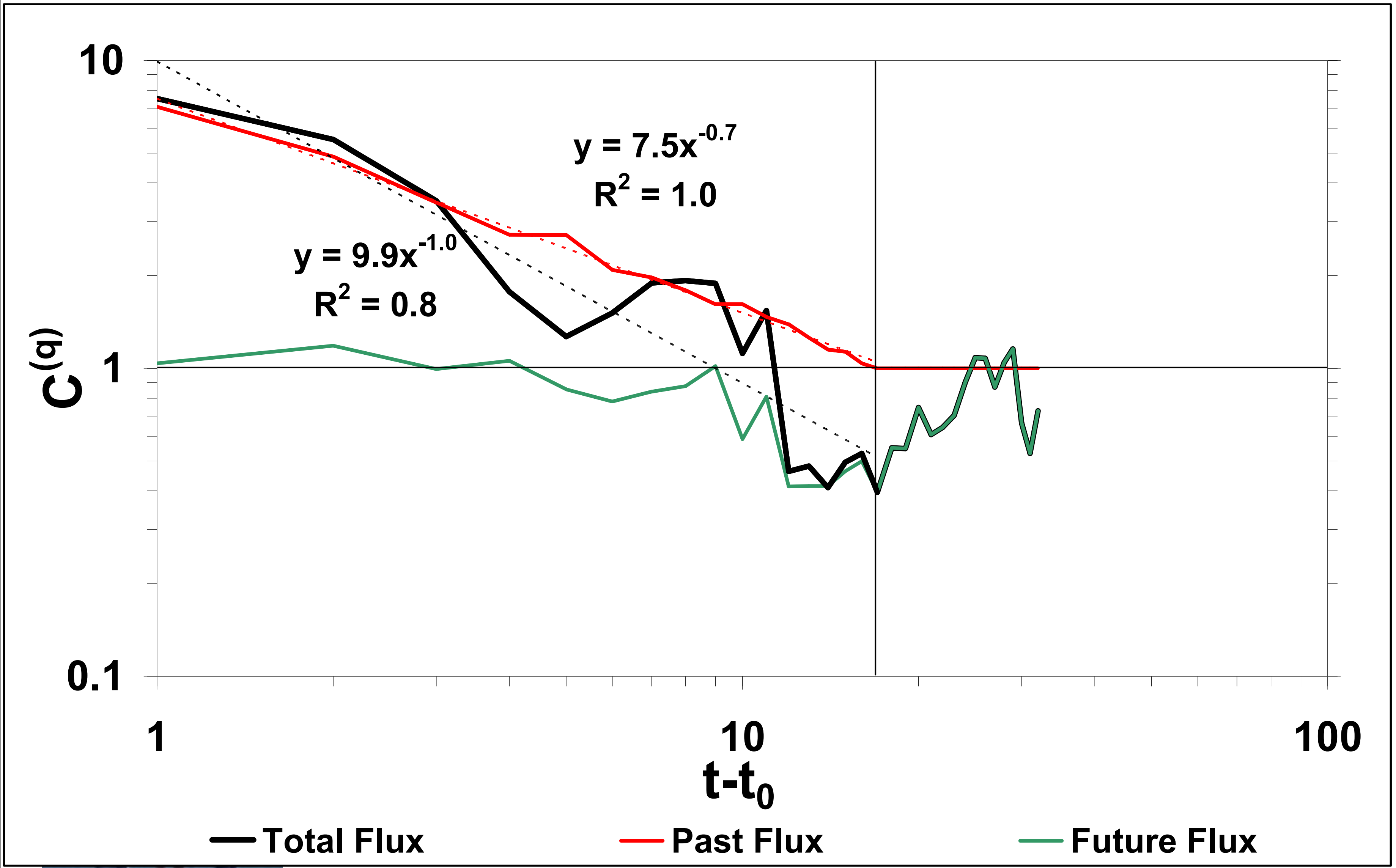
Realisations A,B,C (252^2) have common past ($t=0$, $t_0=32$) for $t= 0, 64$:

A B are 2 stochastic forecasts: similar complexity

C is a deterministic forecast : relaxation of the past structures, the small scale complexity is lost !



Decay of past information



Correlation analysis of Fluxes:

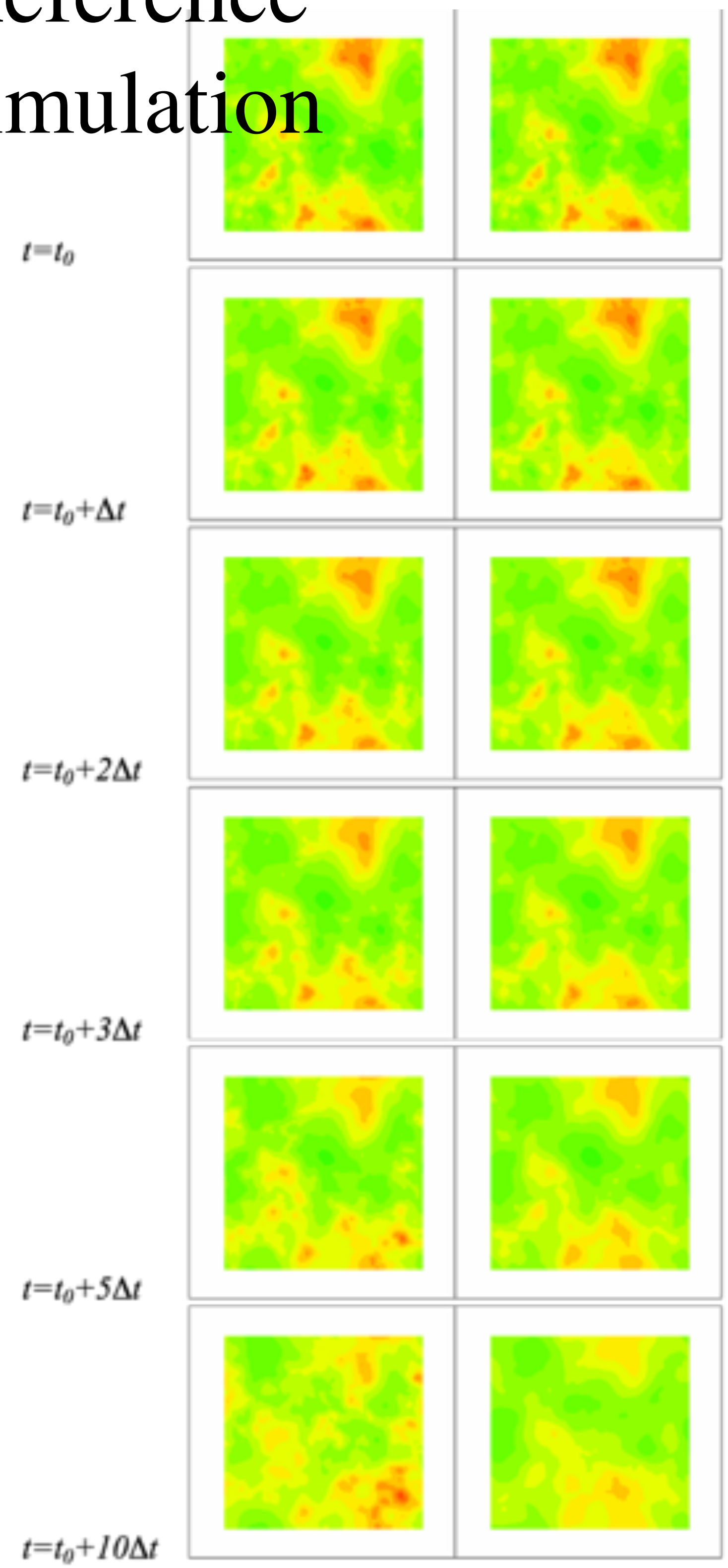
-similar for the total flux and that of the past,

-future flux correlation

- only oscillate
around unity (stochastic conservation).

(naive) ensemble prediction

Reference simulation

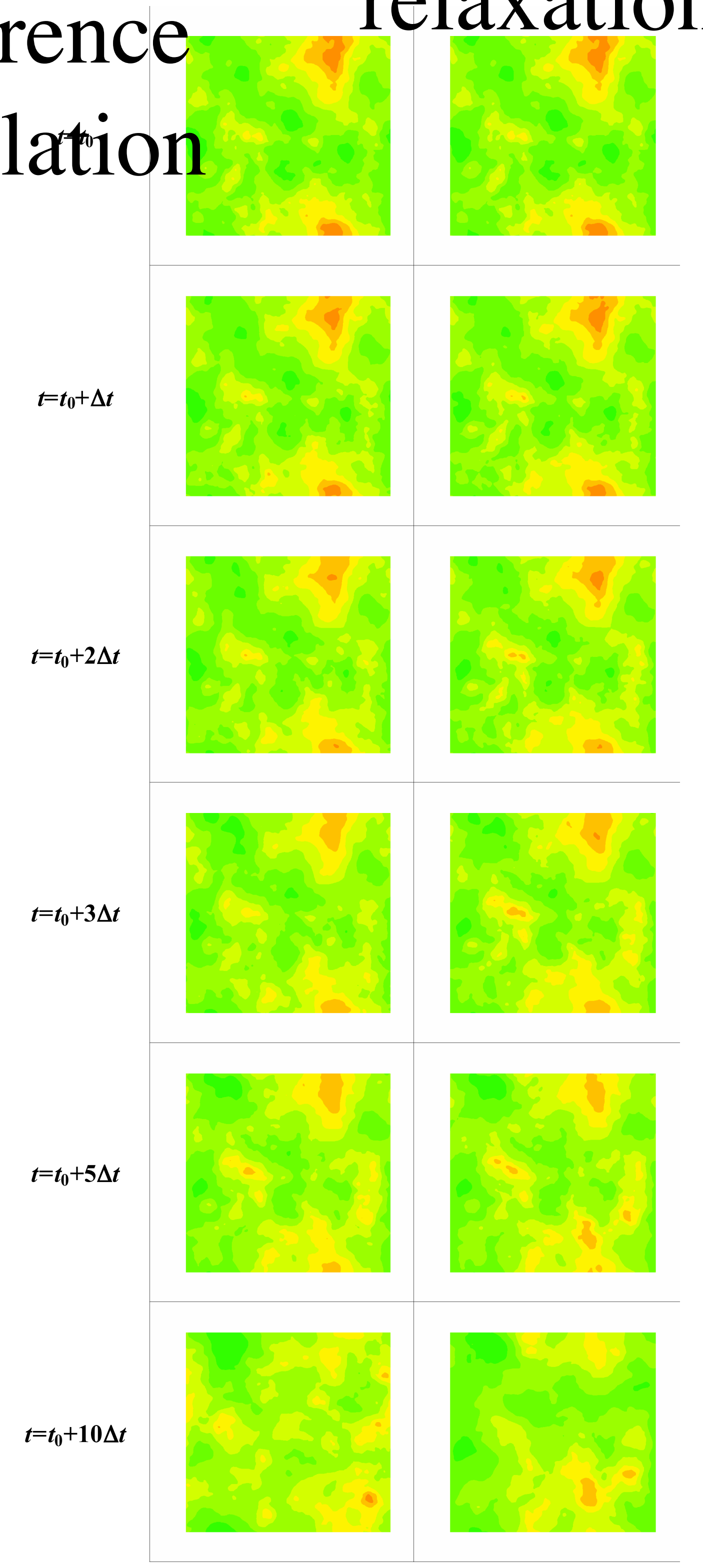


$$\alpha = 1.8,\ C_1 = 0.1,$$
$$H = 1/3,\ \ H_t = 1/3, \lambda = 256$$

Average of 20 forecasts (independent flux subgenerators): still rather blurred…

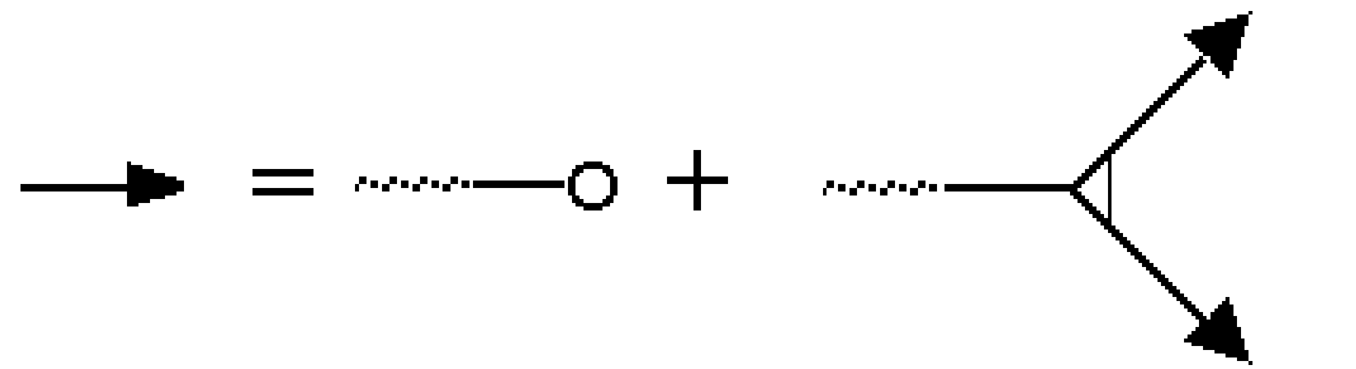
Reference simulation

relaxation

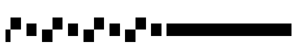


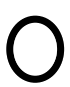
Fundamental problem: nonlinearity

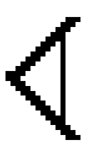
$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot grad(\underline{u}) = \underline{f} - \frac{1}{\rho}\underline{grad}(p) + \nu \Delta \underline{u}$$

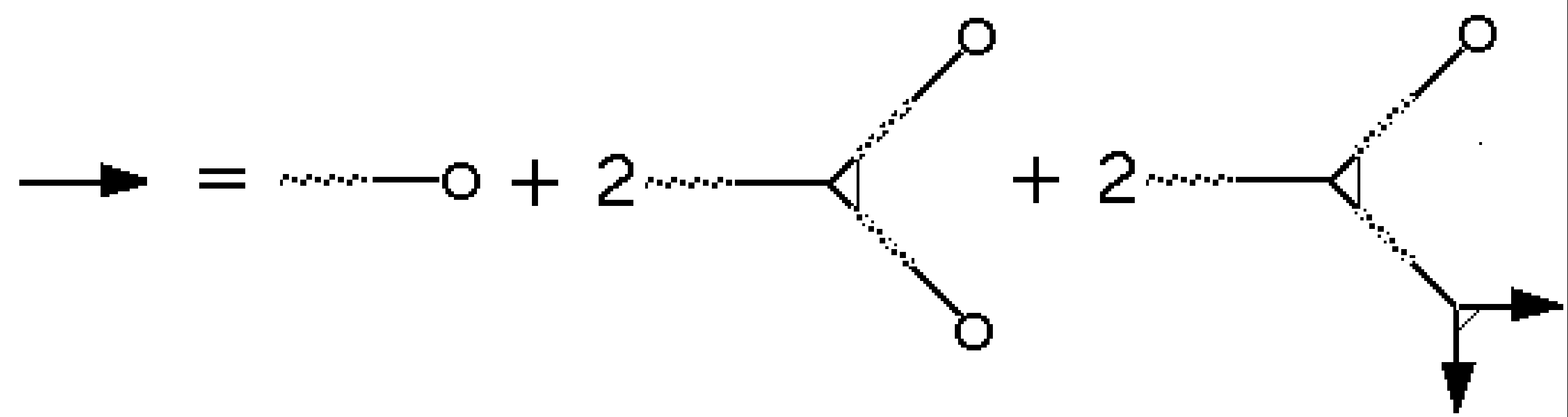


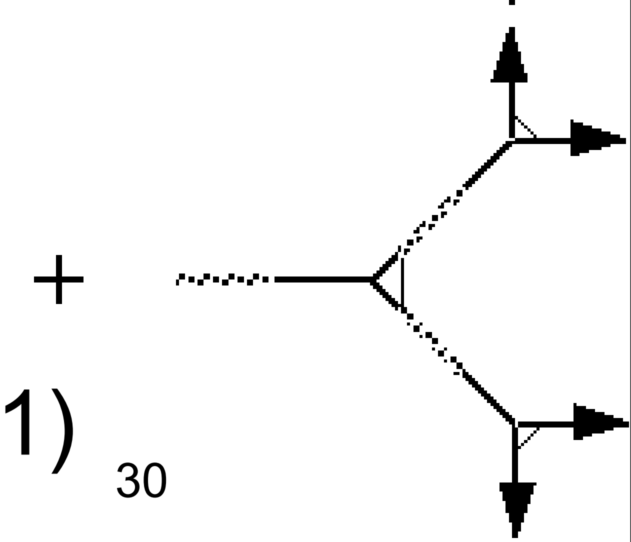
 = $\underline{u}(\underline{x},t)$

 = bare propagator

 = $\underline{f}(\underline{x},t)$

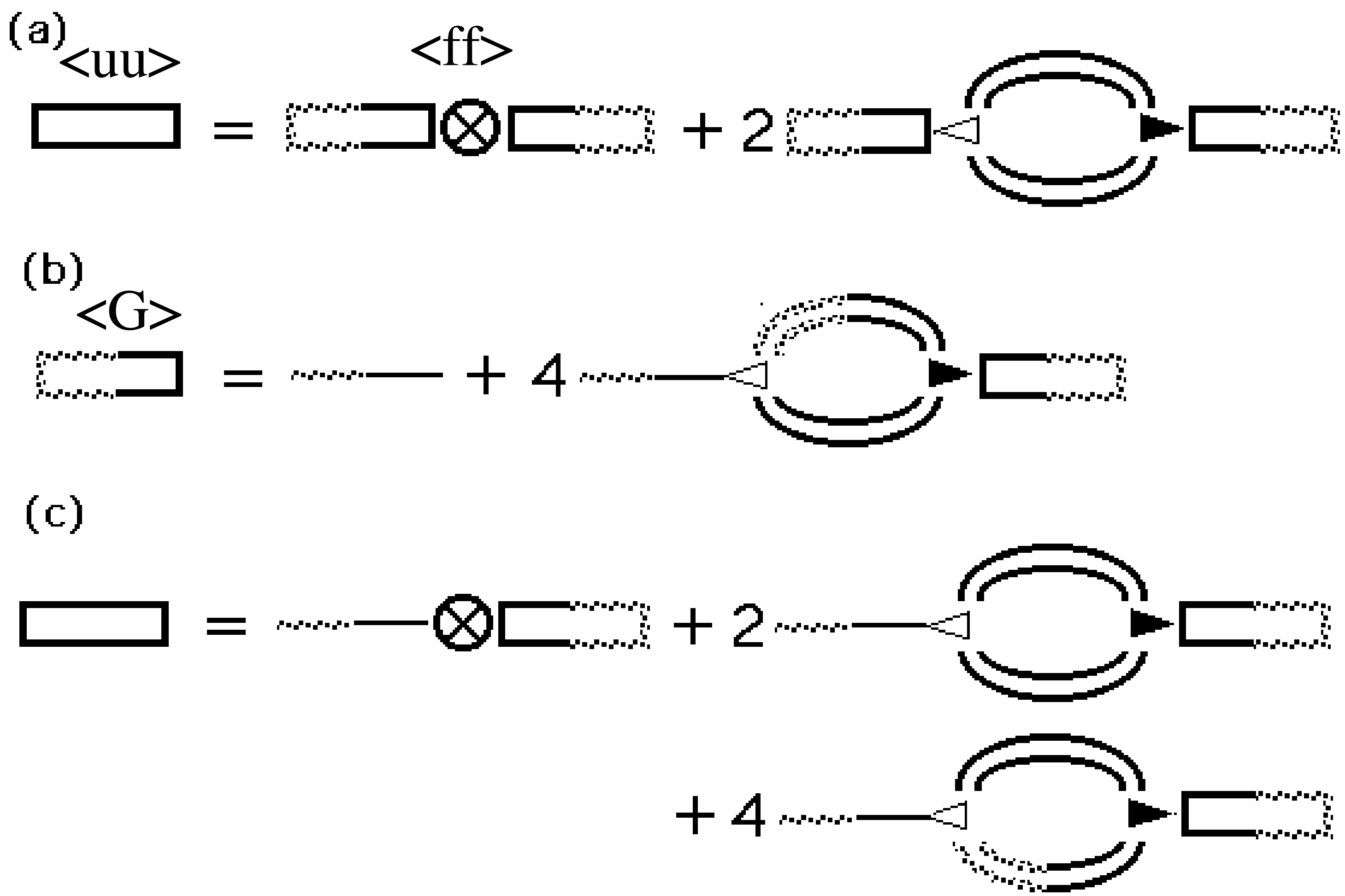
 = vertex



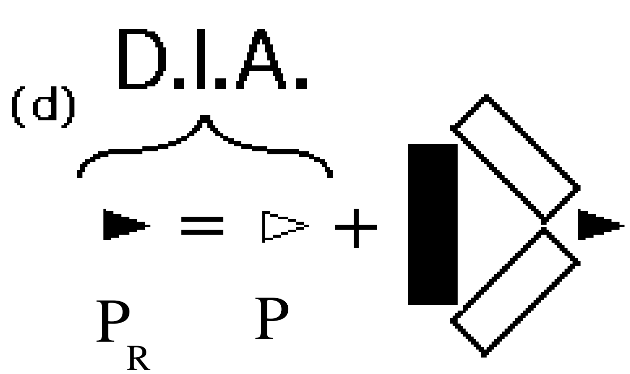


Endless proliferation of higher and higher order diagrams (*Re* >>1)

Quasi-gaussian dead end



G_R renormalized propagator



main assumption:
the forcing *f* is (quasi-) gaussian
however, the renomalization of the vertex
is non trivial and unsolved !
=> fundamental importance of

Fractionnaly Integrated Flux model (FIF, vector version)

FIF assumes that both the renomalized propagator G_R and force f_R are known:

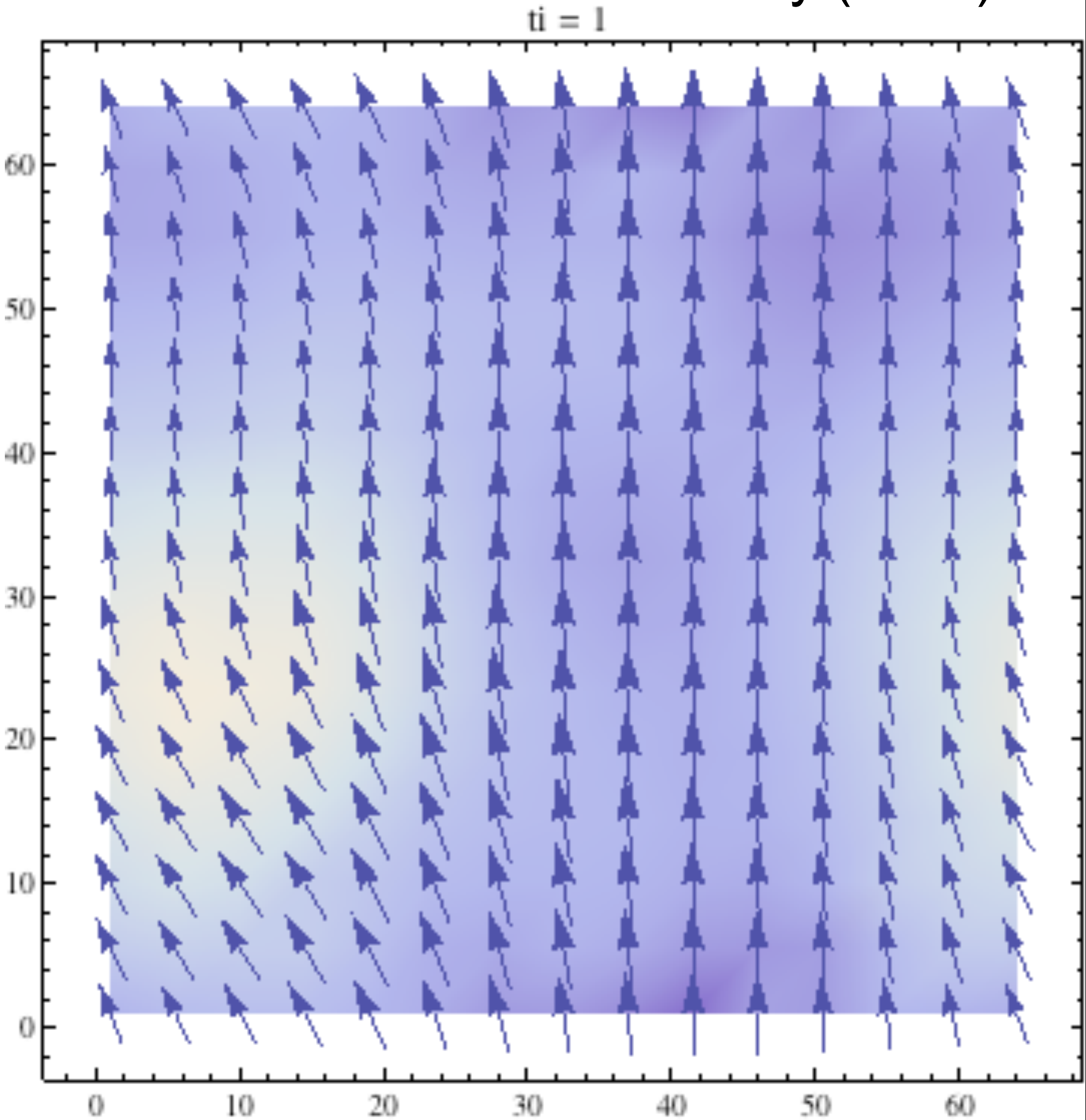
$$G_R^{-1} * u = f_R$$

where: $f_R = \varepsilon^a$

G_R^{-1} is a fractionnal differential operator

$\mathcal{E}$ results from a continuous, vector, multiplicative cascade (Lie cascade)

Complex FIF simulation of a 2D cut of
wind and its vorticity (color)



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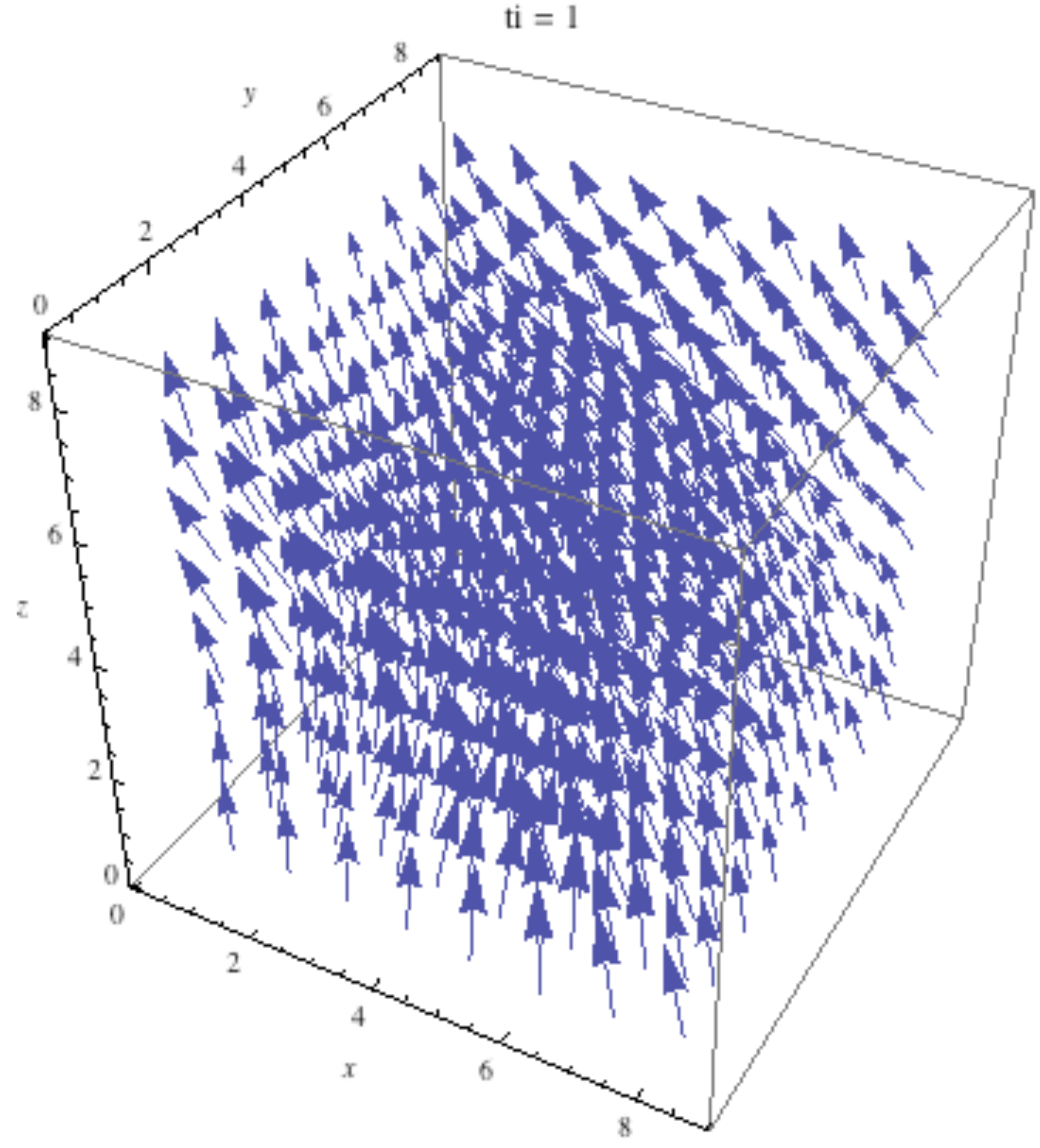
3D FIF wind simulation based on quaternions



Conclusions

- Prediction in space-time complex systems is still at its infancy.
- Requires critical examination of concepts that emerged from the study of systems that are complex only in time (e.g. characteristic predictability time),
- space-time complex systems :
  - Relative space/time symmetry,
  - no characteristic times of predictability.
    - i.e. power-law decays of the predictability
    - higher predictability limits !