ANALYSIS OF KRIGED RAINFIELDS USING MULTIFRACTALS

by

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ABSTRACT

Kriging interpolation is largely used in geostatistics to characterise the spatial structure of data and it is established in general based upon the stationary or intrinsic assumptions; however, the consequence of this second-order approximation is that the local singularities (or extremes) could be smoothed off. This drawback could be magnified as a finer-scale phenomenon is being investigated, such as urban rainfall. Unlike Kriging, the theory multifractals provides a more complete description of the structure of data by considering a range of orders of statistical moments. This work demonstrates the link between multifractal analysis and the Kriging interpolation and finds that Kriging uses only part of information that is included in multifractals. This causes the loss of local singularity of Kriged rainfall field and could be improved by combining it with singularity analysis. A possible solution is proposed in this work and will be implemented and presented in the workshop.

Keywords: multifractals, Kriging, Gaussian

1 INTRODUCTION

Raingauges remain the most reliable sensors providing direct and accurate point rainfall measurements over the ground surface, which are widely used as "ground truth". The major drawback of raingauge data lies in its limited ability to characterise the spatial variation in rainfall. Radar sensors have been widely used to compensate for this drawback, providing high-resolution spatial and temporal rainfall information; however, not every country or city is able to afford radar. Many studies have therefore been conducted, aiming to synthesise the spatial structures of rainfall from point raingauge information using interpolation methods (Tabios et al., 1985; Syed et al., 2003; Looper and Vieux, 2011).

The general idea of interpolation methods is to predict unknown values from data observed at known locations and most of the existing methods are implemented based upon the assumption that each unknown value is a linear combination of known data. The weighing of the linear combination is in general determined according to the spatial association that widely-observed in geo-data, such as rainfall (Oliver and Webster, 1990). Kriging is one of the most largely-used interpolation methods because it produces unbiased (the mean of error is zero) and optimal prediction (the variance of the errors is minimised). Due to these features, Kriging methods are a very popular tool to generate (or predict) spatial estimates from point (or raingauge) rainfall records and consequently provide comparable information that can be further used, for example, to combine with radar or satellite rainfall observations (Todini, 2001; Wang et al., 2012).

However, Kriging methods usually assume that the rainfall fields are Gaussian variables. That means both that the marginal distribution of rainfall at a point is normally distributed, and that the spatial structure of rainfall is well captured by its second-order properties in a covariance matrix. The first assumption is however a poor representation of the distribution of rainfall depths (Wilson et al., 1991), while the second is questionable in view of the scaling properties of rainfall depths (Cheng, 2005).

For non-Gaussian-distributed random fields (such as high-resolution, small-scale rainfall fields over urban catchments), Kriging methods may not be able to interpolate (or predict) rainfall details reliably. Multifractals provide a promising framework to describe highly non-linear processes through scaling analyses, and it has proven to be a useful tool to analyse non-Gaussian random fields (Schertzer and Lovejoy, 1987; Lovejoy and Schertzer, 1990; Deidda et al., 1999).

This work employs the theory of Multifractals to study the key technique –semi-variogram– that is used in the Kriging interpolation. By analysing the relation between variograms and the multifractals, the suitability of using Kriging interpolation techniques for urban rainfall generation will be preliminarily assessed. The possible improvement will be then proposed, implemented and presented in the workshop.

2 METHODOLOGY

2.1 Data set

Composite radar images (with 1 km and 5 min resolutions) of an event (identified as a convective storm) crossing Greater London area on 26th May 2011were used in this analysis. These data are produced by the UK Met Office Nimrod system and under routine quality-control processes (Golding, 1998; Harrison et al., 2009). The image of a 40 km x 40 km snapshot at 15.25 (GMT) of this event is shown in *Figure 2 (a).*

2.2 Variograms and Multifractals

Semi-variogram (or variogram) is a useful to represent the spatial association of geo-data. It shows the dissimilarity between two values at different locations separating by the distance h and can be defined as (Margaret, 1998):

$$\gamma(h) = 0.5Var[Z(\mathbf{x}+h) - Z(\mathbf{x})]\,. \tag{1}$$

This equation can be further simplified, if the assumptions of stationary or intrinsic are applied, as

$$\gamma(h) = 0.5E[\{Z(\mathbf{x}+h) - Z(\mathbf{x})\}^2]\,, \tag{2}$$

where Var[.] and E[.] respectively represent the variance and expectation operators, and $Z(\mathbf{x})$ is a random variable (or rainfall data) at the location $\mathbf{x}$. The variogram models that are commonly used to fit the experimental (or raw) variograms include spherical, exponential and Gaussian models (Margaret, 1998; Chilès and Delfiner, 2012). To link the variogram to the multifractals, a relation was derived by Cheng and Agterberg (1996), expressed as:

$$\gamma_{\varepsilon}(k) = c\varepsilon^{\tau(2)-1}[1 - 0.5\{(k+1)^{\tau(2)+1} - 2k^{\tau(2)+1} + (k-1)^{\tau(2)+1}\}]\,, \tag{3}$$

where ε is the (spatial) measuring scale of data (i.e. sidelength of radar grid in this work); k is equal to the h/ε; $\tau(q)$ is the Legendre transform of multifractal spectrum $f(\alpha)$ and $\tau(2)$ is related to the second-order statistical moment (or variance) of data; and c is a to-be-determined coefficient that can be obtained by fitting to experimental (or raw) variograms. Eq. (3) (denoted MF variogram model in the following context) demonstrates that, as compared to the multifractals which uses the $\tau(q)$ curve over a range of q to characterise the spatial structure of values, variograms use only the $\tau(2)$ feature (i.e. second-order property) to characterise spatial values. In addition, from the MF model, it can be seen that variogram can be a function of measuring scale ε; if c is determined at a certain scale (e.g. 4-km), in theory the variograms at other scales (e.g. 2-, 1-km and 500-m) could be estimated (see *Figure 1* (right) as an example).

The widely-used Gaussian variogram model is employed here to compare with the MF variogram model:

$$\gamma(h) = p + \omega(1 - e^{-(h/a)^2})\,, \tag{4}$$

where p is the used to represent nugget effect (i.e. the discontinuity close to origin), and ω is the limit of a variogram called sill at the distance a (called range). *Table I* gives the estimates of Gaussian and MF variogram models for 26/05/2011 15.25 radar image at 1-km spatial scale. It can be seen in *Figure 1* that, unlike the Gaussian model which show quadratic (highly continuous) behaviour in the vicinity of the origin, the MF variogram model show relatively linear behaviour within the short distance, which is similar to the spherical and exponential models.

Table I – Estimates of Gaussian and MF semi-variogram parameter values for the radar rainfall image at 15.25 (GMT) on 26/05/2011

Gaussian	Nugget (p)	Sill (ω)	Range (a)
	0.162	8.866	3889.28
MF	Constant (C)	$\tau(2)$	Scale (ε)
	58.85	0.804	1000.0

2.3 Kriging interpolation

Based upon the MF variogram shown in *Figure 1*, the Kriged rainfall field of the radar image 26/05/2011 15.25 can be obtained and shown in *Figure 2 (b).* It can be seen that in general Kriging interpolation well synthesise the shape of storm, but relatively smooth and symmetric. From the statistics of these images, it can be found although the mean rainfall rates over whole image is preserved, the rainfall maximum is underestimated and the overall spatial structure (i.e. standard deviation, Std) is smoothed off. This demonstrates the deficiency of using second-order statistical properties only to approximate (or to Krig) rainfall structures.

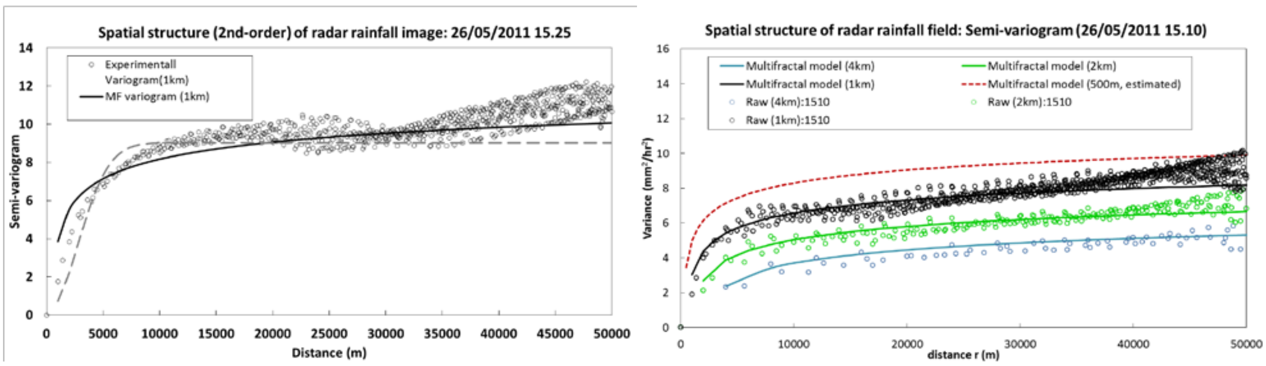


Figure 1 – Experimental (raw) variograms and the associated variogram models: (left) the raw variogram (the round markers) of the radar image measured at 15.25 GMT on 26/05/2011and the associated Gaussian (the grey dashed line) and MF (the dark solid line) variogram models; (right) the raw variogram (the round markers) of the 15.10 radar image at different scales (4-, 2- and 1-km) and the estimated 2-, 1-km and 500-m MF models which are fitted using 4-km data.

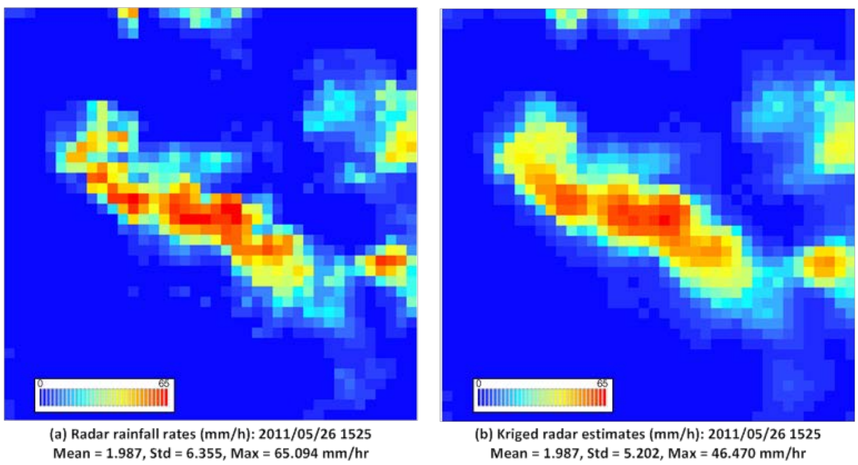


Figure 2 – A snapshot of (a) radar measurements at 15.25 GMT for the 26/05/2011 event crossing Greater London area, and (b) the associated Kriged rainfall field using MF variogram model

2.4 Singularity

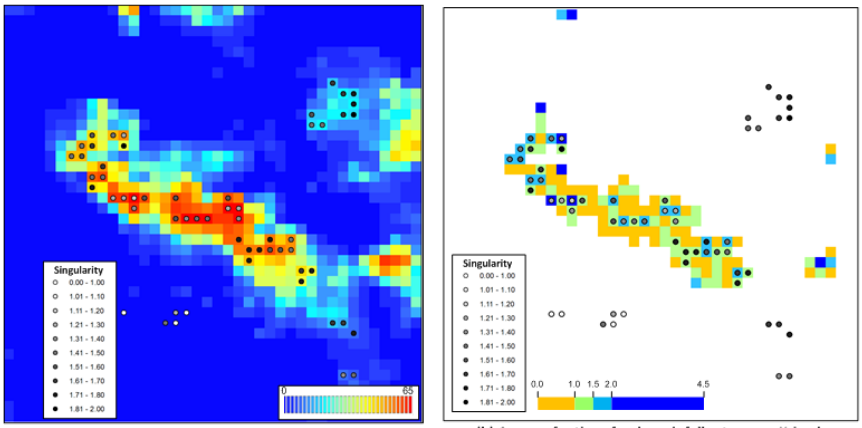
The singularity, in the context of multifractals, is an index used to characterise the variation of statistical behaviour of data values as the measuring scale changes. The removal of singularity could be critical for small-scale applications (e.g. urban flood modelling) because the distribution of singularities is usually consistent with the distribution of the anomalies of singular physical processes (e.g. rainfall) that result in anomalous amounts of energy releases at a fine (spatial and temporal) scale (Schertzer and Lovejoy, 1987). However, in the process of (Kriging) interpolation, the singularity of data could be smoothed off in order to obtain more robust spatial association, and consequently some valuable information of local variability is removed.

In order to take into account the singularity that could be smoothed off by the interpolation process, a more general form of data values proposed by Cheng et al. (1994) is employed here, i.e.,

$$Z(\mathbf{x}, \varepsilon) = c(\mathbf{x})\varepsilon^{\alpha(\mathbf{x})-E}, \tag{5}$$

where $c(\mathbf{x})$ is a constant data value at locations $\mathbf{x}$ and is invariant as measuring scale ε changes; $\alpha(\mathbf{x})$ is the singularity index and E is the Euclidean dimension (E = 2 for plane data). If data values do not show singularity, $\alpha(\mathbf{x})$ is equal to E, and consequently the average of data values within the $\varepsilon \times \varepsilon$ area retains the same as scale changes (i.e. $Z(\mathbf{x}, \varepsilon) = c(\mathbf{x})$). To implement this in rainfall images, the mean rainfall rates within the boxes with variable sidelengths ε's (i.e. 1, 3, 5, 7, 9 km in this work) are firstly computed. The logarithms of these rates and the associated sidelengths are then compared. If a well linear relation can be observed, it means the scaling is followed and the singularity $\alpha(\mathbf{x})$ of the dataset can be derived.

Figure 3 shows the mapping between the singularity values and radar rainfall rates and the mapping between the singularity values and the ratios of radar rainfall rates over Kriged rainfall estimates. It can be seen that the locations of rainfall peaks and the rainfall rates that are smoothed off by the Gaussian approximation are highly consistent with the occurrence of singularities. This demonstrates that the Kriging interpolation smoothens off the local singularity, which decreases its suitability for urban-scale applications; however, this also shows the potential to restore the local extremes that are smoothed off in the Kriged rainfall field by taking into account the singularity, and therefore the suitability of Kriging interpolation to urban-scale applications can be improved.



(a) A map of radar rainfall rates (in mm/h) and the singularity values

(b) A map of ratios of radar rainfall rates over Kriged estimates and the estimated singularity values

Figure 3 – Gaussian and MF semi-variograms compared to empirical variogram estimated from radar observations

3 CONCLUSIONS

In this work, the key technique –semi-variogram– of Kriging interpolation that is analysed using multifractals. It can be found that whereas the multifractals characterises spatial rainfall data using a range of statistic moments, Kriging interpolation only uses part of the statistical moments (i.e. the second-order property, $\tau(2)$) to approximate the complex structure of radar rainfall data. This may smooth off the local singularity values (or extremes), which are however critical for urban-scale hydrological applications. A new interpolation form was proposed by Cheng (2005) to take into account local singularity, expressed as:

$$Z(\mathbf{x}_0,\varepsilon)=\sum_{\mathbf{x}_i\in\Omega(\mathbf{x}_0,\varepsilon)}\lambda_i(h(\mathbf{x}_i,\mathbf{x}_0))\varepsilon^{\alpha(\mathbf{x}_0)-\alpha(\mathbf{x}_1)}Z(\mathbf{x}_i,\varepsilon)\,, \tag{6}$$

where $\alpha(\mathbf{x}_0)$ and $\alpha(\mathbf{x}_i)$ are respectively the singularity indices at location of interest $\mathbf{x}_0$ and the neighbourhood points $\mathbf{x}_i$; and λ_i is the weighing used to linearly combine the surrounding data, which is a function of the distance h between $\mathbf{x}_0$ and each neighbourhood point $\mathbf{x}_i$ and can be derived from variogram. This new interpolation form has shown the potential to improve the conventional Kriging interpolation in terms of the presence of local singularity. However, the scaling feature of variogram (demonstrated in Eq. (3) and *Figure 1* (right)), which will consequently affects the weighting in Eq. (6), was not considered in this new interpolation form. This may not satisfactorily reflect the variation between spatial data as a finer-scale spatial structure is being explored. An improved version of Eq. (6), considering the scaling of variogram, will be implemented and presented in the workshop.

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